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A DIRICHLET PROCESS CHARACTERIZATION OF A CLASS OF REFLECTED DIFFUSIONS

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For a class of stochastic differential equations with reflection for which a certain $\mathbb{L}^p$ continuity condition holds with $p > 1$, it is shown that any weak solution that is a strong Markov process can be decomposed into the sum of a local martingale and a continuous, adapted process of zero p-variation. When $p = 2$, this implies that the reflected diffusion is a Dirichlet process. Two examples are provided to motivate such a characterization. The first example is a class of multidimensional reflected diffusions in polyhedral conical domains that arise as approximations of certain stochastic networks, and the second example is a family of two-dimensional reflected diffusions in curved domains. In both cases, the reflected diffusions are shown to be Dirichlet processes, but not semimartingales.

1. Introduction.

1.1. *Background and motivation.* This work identifies fairly general sufficient conditions under which a reflected diffusion can be decomposed as the sum of a continuous local martingale and a continuous adapted process of zero p-variation, for some p greater than one. As motivation for such a characterization, two examples of classes of reflected diffusions are considered. The first example consists of a large class of multidimensional, obliquely reflected diffusions in polyhedral domains that arise in applications. Reflected diffusions in this class are shown not to be semimartingales, but to belong to the class of so-called Dirichlet processes. Dirichlet processes are processes that can be expressed (uniquely) as the sum of a local martingale and a continuous process that has zero quadratic variation, and thus correspond

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to the case when $p = 2$ in the decomposition mentioned above. The second example consists of a class of two-dimensional reflected diffusions in curved "valley-shaped" domains that were first considered by Burdzy and Toby in [3]. Once again, the reflected diffusion is shown to admit a decomposition of the type mentioned above, but in this case the magnitude of p depends, in a sense made precise in the sequel, on the curvature of the domain.

Processes that admit a decomposition of the type mentioned above are clearly an extension of the class of continuous semimartingales. As is well known, semimartingales form an important class of processes for stochastic integration, they are stable under $\mathcal{C}^2$ transformations and admit an Itô change-of-variable formula. However, there are many natural operations that lead out of the class of semimartingales and motivate the consideration of Dirichlet processes. For example, $\mathcal{C}^1$ functionals of Brownian motion, certain functionals of symmetric Markov processes associated with Dirichlet forms [17], and Lipschitz functionals of a broad class of semimartingale reflected diffusions in bounded domains [26, 27], are all Dirichlet processes that are in general not semimartingales. Moreover, Dirichlet processes exhibit many nice properties analogous to semimartingales. They admit a natural, Doob–Meyer-type decomposition [5], they are stable under $\mathcal{C}^1$ transformations (see Proposition 11 of [28] and also [1]) and there are extensions of stochastic calculus and Itô's formula that apply to Dirichlet processes (see [12, 14] and Chapter 4 of [28]) or, more generally, to processes that admit a decomposition as the sum of a local martingale and a continuous, adapted process of bounded p-variation, for $p \in (1, 2)$ [1]. Furthermore, the theory of rough paths (see, e.g., [16] or [21]) applies to processes whose paths have bounded p-variation for an arbitrary $p \in [1, \infty)$.

The theory of reflected diffusions is most well-developed for semimartingale or symmetric reflected diffusions. In particular, the Skorokhod problem approach to the study of reflected diffusions [8, 22, 29] is automatically limited to semimartingales, while the Dirichlet form approach is best suited to analyze symmetric diffusions (see, e.g., [4, 17]). However, using the submartingale formulation of Stroock and Varadhan [30] or the extended Skorokhod problem [22], it is possible to construct reflected diffusions that are neither semimartingales nor symmetric processes [2, 3, 23, 24, 31]. This leads naturally to the question of determining when these reflected diffusions are semimartingales and, if they are not semimartingales, whether they belong to some other tractable class of processes such as Dirichlet processes. There has been a substantial body of work that shows that, under certain conditions on the domain and reflection directions (namely, the completely-$\mathcal{S}$ condition and generalizations of it), the associated reflected diffusion is a semimartingale [22, 32]. In contrast, it has been a longstanding open problem (see Section 4(iii) of [32]) to develop a theory for multidimensional reflected diffusions for which this condition fails to hold (some results in two

dimensions can be found in [2, 3, 31]). As shown in [23, 24], such reflected diffusions arise naturally as approximations of a so-called generalized processor sharing model used in telecommunication networks. Thus, the development of such a theory is also of interest from the perspective of applications.

The first main result of this work (Theorem 3.1) shows that multidimensional reflected diffusions that belong to a slight generalization of the family of reflected diffusions obtained as approximations in [23, 24] fail to be semimartingales. In two dimensions and for the case of reflected Brownian motion, this result follows from Theorem 5 of [31] (also see [2] for an alternative proof of this result). However, the analysis in [31] uses constructions in polar coordinates that appear not to be easily generalizable to higher dimensions. We follow a different approach, which is independent of dimension and which allows us to establish the result for uniformly elliptic reflected diffusions, with possibly state-dependent diffusion coefficients, rather than just reflected Brownian motion.

The next main result (Theorem 3.5) shows that reflected diffusions that belong to a broad class admit a decomposition as the sum of a local martingale and a process of zero p-variation, for some $p > 1$. This class consists of weak solutions to stochastic differential equations with reflection that are Markov processes and have locally bounded drift and dispersion coefficients and satisfy a certain L^p continuity requirement (see Assumption 2). This continuity requirement is satisfied, for example, when the associated extended Skorokhod map is Hölder continuous, but also holds under much weaker conditions that do not even require that the (extended) Skorokhod map be well-defined (see Remark 2.4). When the extended Skorokhod map is well-defined and Lipschitz continuous, this implies that the associated reflected diffusion is a Dirichlet process. Using the latter result, it is shown in Corollary 3.6 that the nonsemimartingale reflected diffusions considered in Theorem 3.1 are Dirichlet processes. Our next result concerns the class of reflected Brownian motions introduced in [3], which were shown in [2] not to be semimartingales. In Corollary 3.7, Theorem 3.5 is applied to show that even in cusplike domains, the associated reflected Brownian motions are Dirichlet processes, thus partially resolving an open question raised in [3].

The paper is organized as follows. Some common notation used throughout the paper is first summarized in Section 1.2. The class of stochastic differential equations with reflection under consideration, and the related motivating examples, are introduced in Section 2. Section 3 contains a rigorous statement of the main results; the proof of Theorem 3.1 is presented in Section 4, while the proofs of Theorem 3.5 and Corollary 3.6 are given in Section 5. Some elementary results required in the proofs are relegated to the Appendix.

1.2. *Notation.* As usual, $\mathbb{R}_+$ or $[0,\infty)$ denote the space of all nonnegative reals, and $\mathbb{N}$ denotes the space of all positive integers. Given two real numbers a and b, $a\wedge b$ and $a\vee b$ denote the minimum and maximum, respectively, of a and b. For each positive integer $J\geq 1$, $\mathbb{R}^J$ denotes J-dimensional Euclidean space and the nonnegative orthant in this space is denoted by $\mathbb{R}^J_+=\{x\in\mathbb{R}^J : x_i\geq 0 \text{ for } i=1,\dots,J\}$. The Euclidean norm of $x\in\mathbb{R}^J$ is denoted by $|x|$ and the inner product of $x,y\in\mathbb{R}^J$ is denoted by $\langle x,y\rangle$. The vectors $(e_1,e_2,\dots,e_J)$ represent the usual orthonormal basis for $\mathbb{R}^J$, with e_i being the ith coordinate vector. Given a vector $u\in\mathbb{R}^J$, u^T denotes its transpose, with analogous notation for matrices For $x,y\in\mathbb{R}^J$ and a closed set $A\subset\mathbb{R}^J$, $d(x,y)$ denotes the Euclidean distance between x and y, and $d(x,A)=\inf_{y\in A} d(x,y)$ denotes the distance between x and the set A. For each $r\geq 0$, $N_r(A)=\{x\in\mathbb{R}^J : d(x,A)\leq r\}$. The unit sphere in $\mathbb{R}^J$ is represented by $S_1(0)$. Given a set $A\subset\mathbb{R}^J$, A° denotes its interior, $\overline{A}$ its closure and ∂A its boundary.

The space of continuous functions on $[0,\infty)$ that take values in $\mathbb{R}^J$ is denoted by $\mathcal{C}[0,\infty)$, and, given a set $G\subset\mathbb{R}^J$, $\mathcal{C}_G[0,\infty)$ denotes the subset of functions f in $\mathcal{C}[0,\infty)$ such that $f(0)\in G$. The spaces $\mathcal{C}[0,\infty)$ and $\mathcal{C}_G[0,\infty)$ are assumed to be equipped with the topology of uniform convergence on compact sets. Given $f\in\mathcal{C}[0,\infty)$ and $T\in[0,\infty)$, $\mathrm{Var}_{[0,T]}f$ denotes the $\mathbb{R}_+\cup\{\infty\}$-valued number that equals the variation of f on $[0,T]$. Also, given a real-valued function f on $[0,\infty)$, its oscillation is defined by

$$Osc(f;[s,t])=\sup_{s\leq u_1\leq u_2\leq t}|f(u_2)-f(u_1)|;\qquad 0\leq s\leq t<\infty.$$

For each $A\in\mathbb{R}^J$, $\mathbb{I}_A(\cdot)$ denotes the indicator function of the set A, which takes the value 1 on A and 0 on the complement of A.

Given two random variables $U^{(i)}$ defined on a probability space $(\Omega^{(i)},\mathcal{F}^{(i)},\mathbb{P}^{(i)})$ and taking values in a common Polish space S, $i=1,2$, the notation $U^{(1)}\overset{(d)}{=}U^{(2)}$ will be used to imply that the random variables are equal in distribution. Given a sequence of S-valued random variables $\{U^{(n)},n\in\mathbb{N}\}$ and U, with $U^{(n)}$ defined on $(\Omega^{(n)},\mathcal{F}^{(n)},\mathbb{P}^{(n)})$ and U defined on $(\Omega,\mathcal{F},\mathbb{P})$, $U^{(n)}\Rightarrow U$ is used to denote weak convergence of the sequence $U^{(n)}$ to U. Also, if the sequence of random variables are all defined on the same probability space $(\Omega,\mathcal{F},\mathbb{P})$, the notation $U^{(n)}\overset{(\mathbb{P})}{\to}0$ is used to denote convergence in probability.

2. The class of reflected diffusions. The class of stochastic differential equations with reflection under study are introduced in Section 2.1, and the basic assumptions are stated in Section 2.2. Some useful ramifications of the assumptions and a motivating example are then presented in Section 2.3.

2.1. *Stochastic differential equations with reflection.* The so-called extended Skorokhod problem (ESP), introduced in [22], is a convenient tool for the pathwise construction of reflected diffusions. The data associated with an ESP is the closure G of an open, connected domain in $\mathbb{R}^J$ and a set-valued mapping $d(\cdot)$ defined on G such that $d(x) = \{0\}$ for $x \in G^\circ$, $d(x)$ is a nonempty, closed and convex cone in $\mathbb{R}^J$ with vertex at the origin for every $x \in \partial G$ and the graph of $d(\cdot)$ is closed. Roughly speaking, given a continuous path ψ, the ESP associated with $(G, d(\cdot))$ produces a constrained version ϕ of ψ that is restricted to live within the domain G by adding to it a "constraining term" η whose increments over any interval lie in the closure of the convex hull of the union of the allowable directions $d(x)$ at the points x visited by ϕ during this interval. We now state the rigorous definition of the ESP. (In [22], the ESP was formulated more generally for càdlàg paths, but the formulation below will suffice for our purposes since we consider only continuous processes.)

Definition 2.1 (Extended Skorokhod problem). Suppose $(G, d(\cdot))$ and $\psi \in \mathcal{C}_G[0,\infty)$ are given. Then $(\phi, \eta) \in \mathcal{C}_G[0,\infty) \times \mathcal{C}[0,\infty)$ are said to solve the ESP for ψ if $\phi(0) = \psi(0)$, and if for all $t \in [0,\infty)$, the following properties hold:

(1) $\phi(t) = \psi(t) + \eta(t)$;
(2) $\phi(t) \in G$;
(3) for every $s \in [0, t)$

$$\eta(t) - \eta(s) \in \overline{\mathrm{co}}\Bigg[\bigcup_{u \in (s,t]} d(\phi(u))\Bigg], \tag{2.1}$$

where $\overline{\mathrm{co}}[A]$ represents the closure of the convex hull generated by the set A.

If (ϕ, η) is the unique solution to the ESP for ψ, then we write $\phi = \bar{\Gamma}(\psi)$, and refer to $\bar{\Gamma}$ as the extended Skorokhod map (ESM).

If a unique solution to the ESP exists for all $\psi \in \mathcal{C}_G[0,\infty)$, then the associated ESM $\bar{\Gamma}$ is said to be well-defined on $\mathcal{C}_G[0,\infty)$. In this case, it is easily verified (see Lemma A.1) that if $\phi = \bar{\Gamma}(\psi)$, then for any $s \in [0,\infty)$, $\phi^s = \bar{\Gamma}(\psi^s)$, where for $t \in [0,\infty)$,

$$\psi^s(t) \doteq \phi(s) + \psi(s+t) - \psi(s), \qquad \phi^s(t) \doteq \phi(s+t). \tag{2.2}$$

Moreover, a well-defined ESM is said to be Lipschitz continuous on $\mathcal{C}_G[0,\infty)$ if for every $T < \infty$, there exists $\overline{K}_T < \infty$ such that, given $\psi^{(i)} \in \mathcal{C}_G[0,\infty)$ with corresponding solution $(\phi^{(i)}, \eta^{(i)})$ to the ESP, for $i = 1, 2$, we have

$$\sup_{s \in [0,T]} |\phi^{(1)}(s) - \phi^{(2)}(s)| \le \overline{K}_T \sup_{s \in [0,T]} |\psi^{(1)}(s) - \psi^{(2)}(s)|. \tag{2.3}$$

The ESP is a generalization of the so-called Skorokhod Problem (SP) introduced in [29]. Unlike the SP, the ESP does not require that the constraining term η have finite variation on bounded intervals (compare Definitions 1.1 and 1.2 of [22]). The ESP can be used to define solutions to stochastic differential equations with reflection (SDERs) associated with a given pair $(G, d(\cdot))$ and drift and dispersion coefficients $b:\mathbb{R}^J \mapsto \mathbb{R}^J$ and $\sigma:\mathbb{R}^J \mapsto \mathbb{R}^J \times \mathbb{R}^N$.

Definition 2.2. Given $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$, the triple $(Z_t, B_t), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$ is said to be a weak solution to the associated SDER if and only if:

(1) $\{\mathcal{F}_t\}$ is a filtration on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies the usual conditions;
(2) $\{B_t, \mathcal{F}_t\}$ is an N-dimensional Brownian motion.
(3) $\mathbb{P}(\int_0^t |b(Z(s))|\, ds + \int_0^t |\sigma(Z(s))|^2\, ds < \infty) = 1\ \forall t \in [0,\infty)$.
(4) $\{Z_t, \mathcal{F}_t\}$ is a J-dimensional, adapted process such that $\mathbb{P}$-a.s., (Z, Y) solves the ESP for X, where $Y \doteq Z - X$ and

$$X(t) = Z(0) + \int_0^t b(Z(s))\, ds + \int_0^t \sigma(Z(s))\, dB(s) \qquad \forall t \in [0,\infty). \tag{2.4}$$

(5) The set $\{t : Z(t) \in \partial G\}$ has $\mathbb{P}$-a.s. zero Lebesgue measure. In other words, $\mathbb{P}$-a.s.,

$$\int_0^\infty \mathbb{I}_{\partial G}(Z(s))\, ds = 0. \tag{2.5}$$

This is similar to the usual definition for weak solutions for SDEs (see, e.g., Definitions 3.1 and 3.2 of [19]), except that property 4 is modified to define reflection and property 5 captures the notion of "instantaneous" reflection (see, e.g., pages 87–88 of [15]). A strong solution can also be defined in an analogous fashion.

Definition 2.3. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an N-dimensional Brownian motion B on $(\Omega, \mathcal{F}, \mathbb{P})$, Z is said to be a strong solution to the SDER associated with $(G, d(\cdot))$, $b(\cdot)$, $\sigma(\cdot)$ and initial condition ξ if $\mathbb{P}(Z(0) = \xi) = 1$ and properties 3–5 of Definition 2.2 hold with $\{\mathcal{F}_t\}$ equal to the completed and augmented filtration generated by the Brownian motion B.

For a precise construction of the filtration $\{\mathcal{F}_t\}$ referred to in Definition 2.3, see (2.3) of [19]. In what follows, given the constraining process Y in property 4 of Definition 2.2, the quantity L will denote the associated total variation measure: in other words, for $0 \le s \le t < \infty$, we define

$$L(s,t) \doteq \mathrm{Var}_{(s,t]} Y \quad \text{and} \quad L(t) \doteq L(0,t]. \tag{2.6}$$

Observe that the process L in the second definition in (2.6) is $\{\mathcal{F}_t\}$-adapted and takes values in the extended nonnegative reals, $\overline{\mathbb{R}}_+$.

2.2. *Main assumptions.* We now introduce certain basic assumptions on $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ that will be used in this work. In Section 2.3, we provide a concrete motivating example of a family of SDERs that arise in applications which satisfies all the stated assumptions. In Section 2.4, we provide another example of a class of SDERs that satisfy these assumptions. The latter class, which consists of two-dimensional reflected diffusions in curved domains, was first studied by Burdzy and Toby in [3].

The first assumption concerns existence of solutions. General conditions on G and $d(\cdot)$ under which this assumption is satisfied can be found in Lemma 2.6, Theorem 3.3 and Theorem 4.3 of [22].

ASSUMPTION 1. *There exists a weak solution $(Z_t, B_t), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$ to the SDER associated with $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ such that $\{Z_t, \mathcal{F}_t; t \geq 0\}$ is a Markov process under $\mathbb{P}$.*

Next, we impose a kind of $\mathbb{L}^p$-continuity condition on the ESM.

ASSUMPTION 2. *There exist $p > 1, q \geq 2$ and $K_T < \infty$, $T \in (0, \infty)$, such that the weak solution Z to the SDER satisfies, for every $0 \leq s \leq t \leq T$,*

$$\mathbb{E}[|Y(t) - Y(s)|^p | \mathcal{F}_s] \leq K_T \mathbb{E}\Big[\sup_{u \in [s,t]} |X(u) - X(s)|^q | \mathcal{F}_s\Big], \tag{2.7}$$

where X is the process defined by (2.4) and $Y \doteq Z - X$.

REMARK 2.4. Assumption 2 holds under rather mild conditions on the ESP—for example, when the following oscillation inequality is satisfied for any solution (ϕ, η) to the ESP for a given ψ: for every $0 \leq s \leq t < \infty$, there exists $C_{s,t} < \infty$ such that

$$Osc(\phi, [s,t]) \leq C_{s,t} Osc(\psi, [s,t]).$$

In this case, since (Z, Y) solve the ESP for X, we have for $0 \leq s \leq t \leq T$,

$$|Y(t) - Y(s)| \leq Osc(Y, [s,t]) \leq C_{s,t} Osc(X, [s,t]) \leq 2C_T \sup_{u \in [s,t]} |X(u) - X(s)|,$$

where $C_T = \max_{0 \leq s \leq t \leq T} C_{s,t} < \infty$, and so Assumption 2 holds with $p = q = 2$ and $K_T = 4C_T^2$. The oscillation inequality can be shown to hold in many situations of interest (see, e.g., Lemma 2.1 of [32]). If the ESM associated with $(G, d(\cdot))$ is well-defined and Lipschitz continuous on $\mathcal{C}_G[0, \infty)$, then the oscillation inequality is also automatically satisfied, and so Assumption 2 again holds with $p = q = 2$. Furthermore, it is easy to see that if the

ESM is well-defined and Hölder continuous on $\mathcal{C}_G[0,\infty)$ with some exponent $\alpha \in (0,1)$, then Assumption 2 holds for any $p \geq 2/\alpha$ and $q = \alpha p$. An example of such an ESM is provided in Section 2.4 (see also Section 5.3 and, in particular, Remark 5.5).

ASSUMPTION 3. *The coefficients b and σ are locally bounded, that is, they are bounded on every compact subset of G.*

2.3. *A motivating example and ramifications of the assumptions.* We now describe a family of multi-dimensional ESPs $(G, d(\cdot))$ that arise in applications. Fix $J \in \mathbb{N}$, $J \geq 2$. The J-dimensional ESPs in this family have domain $G = \mathbb{R}_+^J$ and a constraint vector field $d(\cdot)$ that is parametrized by a "weight" vector $\alpha = (\alpha_1, \ldots, \alpha_J)$ with $\alpha_i > 0$, $i = 1, \ldots, J$, and $\sum_{i=1}^J \alpha_i = 1$. Associated with each weight vector α is the ESP $(\mathbb{R}_+^J, d(\cdot))$, where for $x \in \partial G = \partial \mathbb{R}_+^J$,

$$d(x) \doteq \left\{ \sum_{i: x_i = 0} \beta_i d_i : \beta_i \geq 0 \right\}$$

with

$$(d_i)_j \doteq \begin{cases} -\dfrac{\alpha_j}{1 - \alpha_i}, & \text{for } j \neq i, \\ 1, & \text{for } j = i, \end{cases}$$

for $i, j = 1, \ldots, J$. Reflected diffusions associated with this family were shown in [23, 24] to arise as heavy traffic approximations of the so-called generalized processor sharing (GPS) model in communication networks (see also [7] and [9]). Indeed, the characterization of this class of reflected diffusions serves as one of the motivations for this work.

Next, we introduce a family of SDERs that is a slight generalization of the class of GPS ESPs.

DEFINITION 2.5. We will say $(G, d(\cdot)), b(\cdot)$ and $\sigma(\cdot)$ define a Class $\mathcal{A}$ SDER if they satisfy the following conditions:

(1) The ESM associated with the ESP $(G, d(\cdot))$ is well-defined and, for every $T < \infty$, is Lipschitz continuous (with constant $K_T < \infty$) on $\mathcal{C}_G[0, T]$.
(2) G is a closed convex cone with vertex at the origin, $\mathcal{V} = \{0\}$ and there exists $\vec{\mathbf{v}} \in G$ such that

$$\langle \vec{\mathbf{v}}, d \rangle = 0 \qquad \text{for all } d \in d(x) \cap S_1(0), x \in \partial G \setminus \{0\};$$

(3) There exists a constant $\tilde{K} < \infty$ such that for all $x, y \in G$,

$$|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \leq \tilde{K}|x - y|$$

and

$$|\sigma(x)| \leq \tilde{K}, \qquad |b(x)| \leq \tilde{K}(1 + |x|).$$

(4) The covariance function $a: G \to \mathbb{R}^J \times \mathbb{R}^J$ defined by $a(\cdot) = \sigma^T(\cdot)\sigma(\cdot)$ is uniformly elliptic, that is, there exists $\lambda > 0$ such that

$$u^T a(x) u \geq \lambda |u|^2 \qquad \text{for all } u \in \mathbb{R}^J, x \in G. \tag{2.8}$$

We expect that the conditions in property 3 can be relaxed to a local Lipschitz and linear growth condition on both b and σ, and the main result can still be proved by using localization along with the current arguments. However, to keep the notation simple, we impose the slightly stronger assumptions above.

REMARK 2.6. ESPs in the GPS family defined above were shown to satisfy properties 1 and 2 (the latter with $\vec{\mathbf{v}} = e_1 + \cdots + e_J$) of Definition 2.5 in Theorem 3.6 and Lemma 3.4 of [22], respectively.

In Theorem 2.7, we summarize some consequences of Assumptions 1–3, and also show that Class $\mathcal{A}$ SDERs satisfy these assumptions. The proof essentially follows from Theorem 4.3 of [22] and Proposition 4.1 of [18]. The following set,

$$\mathcal{V} \doteq \{x \in \partial G : \text{there exists } d \in S_1(0) \text{ such that } \{d, -d\} \subset d(x)\}, \tag{2.9}$$

was shown in [22] to play an important role in the analysis.

THEOREM 2.7. *Suppose $(G, d(\cdot)), b(\cdot)$ and $\sigma(\cdot)$ satisfy Assumptions 1 and 2, and let $(Z_t, B_t), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$ be a weak solution to the associated SDER. Then Z is an $\mathcal{F}_t$-semimartingale on $[0, T_{\mathcal{V}})$, where*

$$T_{\mathcal{V}} \doteq \inf\{t \geq 0 : Z(t) \in \mathcal{V}\}, \tag{2.10}$$

and $\mathbb{P}$-a.s., Z admits the decomposition

$$Z(\cdot) = Z(0) + M(\cdot) + A(\cdot), \tag{2.11}$$

where for $t \in [0, T_{\mathcal{V}})$,

$$M(t) \doteq \int_0^t \sigma(Z(s)) \cdot dB(s), \qquad A(t) \doteq \int_0^t b(Z(s))\, ds + Y(t), \tag{2.12}$$

and Y has finite variation on $[0, t]$ and satisfies

$$Y(t) = \int_0^t \gamma(s)\, dL(s), \tag{2.13}$$

where L is given by (2.6) and $\gamma(s) \in d(Z(s))$, dL-a.e. $s \in [0, t]$. Moreover, if $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ satisfy properties 1 and 3 of Definition 2.5, then they also satisfy Assumption 1, Assumption 2 (with $p = q = 2$) and Assumption 3. In this case, $\{Z_t, \mathcal{F}_t\}$ is in fact the pathwise unique strong solution to the SDER, is a strong Markov process and has $\mathbb{E}[|Z(t)|^2] < \infty$ for every $t \in (0, \infty)$ if $\mathbb{E}[|Z(0|^2] < \infty$.

PROOF. Let X be the process defined by (2.4). Then X is clearly a semimartingale and property 4 of Definition 2.2 shows that $\mathbb{P}$-a.s., $(Z, Z-X)$ satisfy the ESP for X. Moreover, Theorem 2.9 of [22] shows that $Y = Z - X$ has $\mathbb{P}$-a.s. finite variation on any closed sub-interval of $[0, T_{\mathcal{V}})$. This shows that Z is an $\mathcal{F}_t$-semimartingale on $[0, T_{\mathcal{V}})$ with the decomposition given in (2.11)–(2.13), and thus establishes the first assertion of the theorem.

Next, suppose $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ satisfy properties 1 and 3 of Definition 2.5. Then property 3 of Definition 2.5 implies Assumption 3 is satisfied. In addition, by Remark 2.4, property 1 ensures that Assumption 2 holds with $p = q \geq 2$. Moreover, Theorem 4.3 of [22] and Proposition 4.1 of [18] show that, in fact, the associated SDER admits a pathwise unique strong solution Z, which is also a strong Markov process. Thus, Assumption 1 is also satisfied. Hence, we have shown that Assumptions 1–3 hold. The last assertion of the theorem can be established using standard techniques, by a modification of the proof in Theorem 4.3 of [22], in the same manner as this result is proved for strong solutions to SDEs, and so we omit the details of the proof. $\square$

We conclude this section by stating a consequence of property 2 of Definition 2.5 that will be useful in the sequel. Let Γ_1 denote the (extended) Skorokhod map associated with the 1-dimensional (extended) Skorokhod problem with $G = \mathbb{R}_+$ and $d(0) = \mathbb{R}_+$, $d(x) = 0$ if $x > 0$. It is well known (see, e.g., [29] or Lemma 3.6.14 of [19]) that Γ_1 is well-defined on $\mathcal{C}_{\mathbb{R}_+}[0, \infty)$, and in fact has the explicit form

$$\Gamma_1(\psi)(t) = \psi(t) + \sup_{s \in [0,t]} [-\psi(s)] \vee 0. \tag{2.14}$$

LEMMA 2.8. *Suppose that $(G, d(\cdot))$ satisfies property 2 of Definition 2.5. If (ϕ, η) solves the associated ESP for $\psi \in \mathcal{C}_G[0, \infty)$, then $\langle \phi, \vec{\mathbf{v}} \rangle = \Gamma_1(\langle \psi, \vec{\mathbf{v}} \rangle)$.*

The proof of this lemma is exactly analogous to the proof of Corollary 3.5 of [22], and is thus omitted.

2.4. *Another motivating example.* We now describe a family of two-dimensional reflecting Brownian motions (henceforth abbreviated to RBMs) in "valley-shaped" domains with vertex at the origin and horizontal directions of reflection. This family of reflected diffusions, which was first studied in [3], is parameterized by two continuous real-valued functions L and R defined on $[0, \infty)$, with $L(0) = R(0) = 0$ and $L(y) < R(y)$ for all $y > 0$. The associated domain G is then given by

$$G \doteq \{(x, y) \in \mathbb{R}^2 : y \geq 0, L(y) \leq x \leq R(y)\}.$$

Let $\partial^1 G \doteq \{(x,y) \in \partial G \setminus (0,0) : x = L(y)\}$ and, likewise, let $\partial^2 G \doteq \{(x,y) \in \partial G \setminus (0,0) : x = R(y)\}$. Then the reflection vector field is defined by

$$d(x,y) = \begin{cases} (1,0), & (x,y) \in \partial^1 G, \\ (-1,0), & (x,y) \in \partial^2 G, \\ \{v : v_1 \geq 0\}, & (x,y) = (0,0). \end{cases}$$

Thus, there are two opposing, horizontal directions of reflection on the two lateral boundaries, and then an additional vertical reflection direction $(0,1)$ at $(0,0)$ to ensure that the Brownian motion can be constrained within the domain. To conform with the general structure of ESPs, at $(0,0)$ we in fact define $d(\cdot)$ to be the convex cone (which, in this case, equals a half-space) generated by the three directions $(1,0)$, $(-1,0)$ and $(0,1)$. Note that $\mathcal{V} = \{0\}$ for this ESM and, when L and R are linear functions, this reflected diffusion is a special case of the Class $\mathcal{A}$ SDER's introduced in the last section.

It was shown in Theorem 1 of [3] (see also Section 4.3 of [2]) that the ESM $\bar{\Gamma}$ corresponding to $(G, d(\cdot))$ is well-defined and thus, when B is a standard two-dimensional Brownian motion, $Z = \bar{\Gamma}(B)$ is a well-defined reflected Brownian motion starting at $(0,0)$ and is also a Markov process (see Theorem 2 of [3]). In Proposition 4.13 of [2], RBMs in this family were shown not to be semimartingales. As an application of the results of this paper, we show that when L and R are sufficiently regular, Z nevertheless admits a useful decomposition (see Corollary 3.7).

3. Statement of main results.

Theorem 2.7 shows that if $\mathcal{V} = \varnothing$ then Z is a semimartingale. In fact, it was shown in Theorem 1.3 of [22] that when $\mathcal{V} = \varnothing$, the ESM coincides with the SM. The main focus of this work is to understand the behavior of reflected diffusions Z associated with ESPs $(G, d(\cdot))$ for which $\mathcal{V} \neq \varnothing$, with the GPS family being a representative example. In [22], it was shown that for the GPS family of ESPs, Z is a semimartingale until the first time it hits the origin. However, the first result of the present paper (Theorem 3.1) shows that Z is not a semimartingale on $[0, \infty)$.

THEOREM 3.1. *Suppose $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ describe a Class $\mathcal{A}$ SDER. Then the unique pathwise solution Z to the associated SDER is not a semimartingale.*

The proof of Theorem 3.1 is given in Section 4.3. As mentioned in Section 1, for the special case when G is a convex wedge in $\mathbb{R}^2$ and the directions of constraint on the two faces are constant and point at each other, $b \equiv 0$ and σ is the identity matrix (i.e., Z is a reflected Brownian motion), this result follows from Theorem 5 of [31] (with the parameters $\alpha = 1$ and the wedge angle π less than 180° therein). The fact that, when $J = 2$, the reflected

Brownian motion Z defined here is the same as the reflected Brownian motion defined via the submartingale formulation in [31] follows from Theorem 1.4(2) of [22]. This two-dimensional result can also be viewed as a special case of Proposition 4.13 of [2]. However, the proofs in [2] and [31] do not seem to extend easily to higher dimensions. In this paper, we take a different approach that is applicable in arbitrary dimensions and to more general diffusions, in particular providing a different proof of the two-dimensional result mentioned above.

As is well known, when a process is a semimartingale, $\mathcal{C}^2$ functionals of the process can be characterized using Itô's formula. Theorem 3.1 can thus be viewed as a somewhat negative result since it suggests that Class $\mathcal{A}$ reflected diffusions and, in particular, reflected diffusions associated with the GPS family that arise in applications, may not possess desirable properties. However, we show in Corollary 3.6 that these diffusions are indeed tractable by establishing that they belong to the class of Dirichlet processes (in the sense of Föllmer). This follows as a special case of a more general result, which is stated below as Theorem 3.5.

In order to state this result, we first recall the definitions of zero p-variation processes and Dirichlet processes (see, e.g., Theorem 2 of [13]).

Definition 3.2. For $p > 0$, a continuous process A is of zero p-variation if and only if for any $T > 0$,

$$\sum_{t_i \in \pi^n} |A(t_i) - A(t_{i-1})|^p \overset{(\mathbb{P})}{\to} 0 \tag{3.1}$$

for any sequence $\{\pi^n\}$ of partitions of $[0, T]$ with $\Delta(\pi^n) \doteq \max_{t_i \in \pi^n}(t_{i+1} - t_i) \to 0$ as $n \to \infty$. If the process A satisfies (3.1) with $p = 2$, then A is said to be of zero energy.

Definition 3.3. The stochastic process Z is said to be a Dirichlet process if the following decomposition holds:

$$Z = M + A, \tag{3.2}$$

where M is an $\mathcal{F}_t$-adapted local martingale and A is a continuous, $\mathcal{F}_t$-adapted, zero energy process with $A(0) = 0$.

Note that this is weaker than the original definition of a Dirichlet process given by Föllmer [13], which requires that M and A in the decomposition (3.2) be square integrable and that A satisfy $\mathbb{E}[\sum_{t_i \in \pi^n} |A_{t_i} - A_{t_{i-1}}|^2] \to 0$ as $\Delta(\pi^n) \to 0$, rather than satisfy (3.1) with $p = 2$. However, our definition can be viewed as a localized version and coincides with Definition 2.4 of [5] (see also Definition 12 of [28]).

REMARK 3.4. The decomposition of a Dirichlet process Z, into a local martingale and a zero energy process starting at 0, is unique. For any $p>1$ and partition π^n of $[0,T]$,

$$\sum_{t_i\in\pi^n} |A(t_{i+1})-A(t_i)|^p \leq \max_{t_i\in\pi^n} |A(t_{i+1})-A(t_i)|^{p-1} \mathrm{Var}_{[0,T]}(A).$$

Therefore, it follows that if A is continuous and of finite variation, then it is also of zero p-variation, for all $p>1$. In particular, this shows that the class of Dirichlet processes generalizes the class of continuous semimartingales.

THEOREM 3.5. *Suppose $(G,d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ satisfy Assumptions 1 and 3, let Z be an associated weak solution that satisfies Assumption 2 for some $p>1$, and let $Y=Z-X$, where X is defined by (2.4). Then Y has zero p-variation.*

As an immediate consequence of Theorem 3.5, Definition 3.3 and Theorem 2.7, we have the following result.

COROLLARY 3.6. *Suppose $(G,d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ satisfy Assumptions 1 and 3, and also Assumption 2 with $p=2$. Then the associated reflected diffusion is a Dirichlet process. In particular, reflected diffusions associated with Class $\mathcal{A}$ SDERs are Dirichlet processes.*

The next consequence of Theorem 3.5 concerns the class of reflected diffusions described in Section 2.4.

COROLLARY 3.7. *Suppose that L and R are two continuous functions on $[0,y]$ given by*

$$L(y)=-c_L y^{\alpha_L}, \qquad R(y)=c_R y^{\alpha_R}, \qquad y\in[0,\infty), \tag{3.3}$$

for some $\alpha_L,\alpha_R,c_L,c_R\in(0,\infty)$, and let $\alpha=\min(\alpha_L,\alpha_R)$. If $\alpha\geq 1$, then the associated two-dimensional reflected diffusion Z described in Section 2.4 is a Dirichlet process, that is, admits the decomposition $Z=B+A$, where A is a process with zero quadratic variation.

It was shown in [2] that, for every $\alpha>0$, Z is not a semimartingale. In contrast, Corollary 3.7 establishes a positive result in this direction, showing that even when the domain has a cusp-like shape (i.e., corresponding to $\alpha>1$), the reflected diffusion is a Dirichlet process. This partially resolves the open question raised in [3], as mentioned in Section 5.3, when L and R are linear (i.e., when $\alpha_L=\alpha_R=1$) the domain is wedge-shaped and the reflected diffusion Z is associated with a Class $\mathcal{A}$ SDER. In this case, Corollary 3.7

follows from Corollary 3.6. The proof of Corollary 3.7 in the general case is given in Section 5.3. It is natural to expect that the reflected diffusion would also be a Dirichlet process when $\alpha < 1$, since this corresponds to nicer "flatter" domains. However, this does not directly follow from the simple proof of Corollary 3.7 given in Section 5.3 (see Remark 5.4).

4. Reflected diffusions associated with Class $\mathcal{A}$ SDERs. Throughout this section, we will assume that $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ describe a Class $\mathcal{A}$ SDER. Let B be an N-dimensional Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{\mathcal{F}_t\}$ be the right-continuous augmentation of the filtration generated by B (see Definition (2.3) given in [19]). Also, let Z be the pathwise unique strong solution to the associated SDER (which exists by Theorem 2.7), let X be defined by (2.4), let $Y \doteq Z - X$ and let L be the total variation process of Y as defined in (2.6). We use $\mathbb{E}$ to denote expectation with respect to $\mathbb{P}$ and, for $z \in G$, let $\mathbb{P}_z$ (resp., $\mathbb{E}_z$) denote the probability (resp., expectation) conditioned on $Z(0) = z$.

This section is devoted to the proof of Theorem 3.1. The key step is to show that the constraining process Y in the extended Skorokhod decomposition for Z has $\mathbb{P}_0$-a.s. infinite variation. More precisely, let $\vec{\mathbf{v}}$ be the vector that satisfies property 3 of Definition 2.5 and, for any given $\varepsilon \geq 0$, consider the hyperplane

$$H_\varepsilon \doteq \{x \in \mathbb{R}^d : \langle \vec{\mathbf{v}}, x \rangle = \varepsilon\} \cap G, \tag{4.1}$$

and let

$$\tau^\varepsilon \doteq \inf\{t \geq 0 : Z(t) \in H_\varepsilon\}. \tag{4.2}$$

We now state the key result in the proof of Theorem 3.1.

THEOREM 4.1. *There exists $T < \infty$ such that $\mathbb{P}_0(L(T) = \infty) > 0$.*

A somewhat subtle point to note is that Theorem 4.1 does not immediately establish the fact that Z is not a semimartingale because we do not know a priori that, if Z were a semimartingale, then its Doob decomposition must be of the form $Z = M + A$ given in (2.11) and (2.12). However, in Section 4.3 (see Proposition 4.12) we establish that this is indeed the case, thus obtaining Theorem 3.1 from Theorem 4.1. First, in Section 4.1, we establish Theorem 4.1 for the case when $b \equiv 0$. The proof for the general case is obtained from this result via a Girsanov transformation in Section 4.2.

4.1. *The zero drift case.* Throughout this section, we assume $b \equiv 0$ and establish the following result.

PROPOSITION 4.2. *If $b \equiv 0$, then we have*

$$\mathbb{E}_0[e^{-L(\tau^1)}] = 0, \tag{4.3}$$

and hence,

$$L(\tau^1) = \infty, \qquad \mathbb{P}_0\text{-}a.s. \tag{4.4}$$

When combined with Lemma 4.11, which shows that $\mathbb{P}_0(\tau^1 < \infty) = 1$ when $b \equiv 0$, Proposition 4.2 yields Theorem 4.1. The proof of Proposition 4.2 is given in Section 4.1.3. The proof relies on an upper bound for $\mathbb{E}_0[e^{-L(\tau^1)}]$, which is obtained in Section 4.1.1, and some weak convergence results, which are established in Section 4.1.2.

4.1.1. *An upper bound.* To begin with, we use the strong Markov property of Z to obtain an upper bound on $\mathbb{E}_0[e^{-L(\tau^1)}]$. Recall the definition of τ^0 given in (4.2) with $\varepsilon = 0$, noting that $H_0 = \{0\}$ because G is a closed convex cone with vertex at 0. Moreover, for $\varepsilon > 0$, we recursively define two sequences of random times $\{\tau_n^\varepsilon\}_{n\in\mathbb{N}}$ and $\{\alpha_n^\varepsilon\}_{n\in\mathbb{N}}$ as follows: $\alpha_0^\varepsilon \doteq 0$ and for $n \in \mathbb{N}$,

$$\begin{aligned} \tau_n^\varepsilon &\doteq \inf\{t \geq \alpha_{n-1}^\varepsilon : Z(t) \in H_\varepsilon\}, \\ \alpha_n^\varepsilon &\doteq \inf\{t \geq \tau_n^\varepsilon : Z(t) \in H_0\}. \end{aligned} \tag{4.5}$$

Since Z is continuous and H_ε and H_0 are closed, it is clear that τ^0, τ_n^ε and α_n^ε are $\mathcal{F}_t$-stopping times. For conciseness, we will often denote τ_1^ε simply by τ^ε, since this is consistent with the notation of τ^ε given in (4.2).

LEMMA 4.3. *For every $\varepsilon \in (0,1)$,*

$$\mathbb{E}_0[e^{-L(\tau^1)}] \leq \frac{\mathbb{E}_0[\mathbb{P}_{Z(\tau^\varepsilon)}(\tau^0 \geq \tau^1)]}{\mathbb{E}_0[\mathbb{P}_{Z(\tau^\varepsilon)}(\tau^0 \geq \tau^1)] + \mathbb{E}_0[\mathbb{E}_{Z(\tau^\varepsilon)}[(1 - e^{-L(\tau^0)})\mathbb{I}_{\{\tau^0 < \tau^1\}}]]}.$$

PROOF. From the elementary inequality

$$L(\tau^1) \geq \sum_{n=1}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1)),$$

it immediately follows that

$$\mathbb{E}_0[e^{-L(\tau^1)}] \leq \mathbb{E}_0[e^{-\sum_{n=1}^{\infty}(L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1))}]. \tag{4.6}$$

For $n \geq 2$, $\alpha_n^\varepsilon \geq \alpha_1^\varepsilon$ and $\tau_n^\varepsilon \geq \alpha_1^\varepsilon$. Hence, on the set $\{\alpha_1^\varepsilon \geq \tau^1\}$, we have $\alpha_n^\varepsilon \wedge \tau^1 = \tau_n^\varepsilon \wedge \tau^1 = \tau^1$ for every $n \geq 2$. Therefore, the right-hand side of (4.6) can be decomposed as

$$\begin{aligned}\mathbb{E}_0[e^{-\sum_{n=1}^\infty (L(\alpha_n^\varepsilon\wedge\tau^1)-L(\tau_n^\varepsilon\wedge\tau^1))}] &= \mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon\geq\tau^1\}}]\\ &\quad + \mathbb{E}_0[e^{-\sum_{n=1}^\infty (L(\alpha_n^\varepsilon\wedge\tau^1)-L(\tau_n^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}].\end{aligned}$$

Conditioning on $\mathcal{F}_{\alpha_1^\varepsilon}$, using the fact that $\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}$, $L(\alpha_1^\varepsilon\wedge\tau^1)$ and $L(\tau^\varepsilon\wedge\tau^1)$ are $\mathcal{F}_{\alpha_1^\varepsilon}$-measurable, the strong Markov property of Z and the fact that $Z_1(\alpha_1^\varepsilon)=0$, last term above can be rewritten as

$$\begin{aligned}&\mathbb{E}_0[e^{-\sum_{n=1}^\infty (L(\alpha_n^\varepsilon\wedge\tau^1)-L(\tau_n^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}]\\ &\quad= \mathbb{E}_0[\mathbb{E}_0[e^{-\sum_{n=1}^\infty (L(\alpha_n^\varepsilon\wedge\tau^1)-L(\tau_n^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}|\mathcal{F}_{\alpha_1^\varepsilon}]]\\ &\quad= \mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}\mathbb{E}_0[e^{-\sum_{n=2}^\infty (L(\alpha_n^\varepsilon\wedge\tau^1)-L(\tau_n^\varepsilon\wedge\tau^1))}|\mathcal{F}_{\alpha_1^\varepsilon}]]\\ &\quad= \mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}\mathbb{E}_{Z(\alpha_1^\varepsilon)}[e^{-\sum_{n=1}^\infty (L(\alpha_n^\varepsilon\wedge\tau^1)-L(\tau_n^\varepsilon\wedge\tau^1))}]]\\ &\quad= \mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}]\mathbb{E}_0[e^{-\sum_{n=1}^\infty (L(\alpha_n^\varepsilon\wedge\tau^1)-L(\tau_n^\varepsilon\wedge\tau^1))}].\end{aligned}$$

Combining the last two assertions and rearranging terms, we obtain

$$\mathbb{E}_0[e^{-\sum_{n=1}^\infty (L(\alpha_n^\varepsilon\wedge\tau^1)-L(\tau_n^\varepsilon\wedge\tau^1))}] = \frac{\mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon\geq\tau^1\}}]}{1-\mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}]}.$$

Together with (4.6), this yields the inequality

$$\mathbb{E}_0[e^{-L(\tau^1)}] \leq \frac{\mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon\geq\tau^1\}}]}{1-\mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}]}. \tag{4.7}$$

We now show that the upper bound stated in the lemma follows from (4.7). Due to the nonnegativity of $L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1)$ and the strong Markov property of Z, we have

$$\begin{aligned}\mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon\geq\tau^1\}}] &\leq \mathbb{E}_0[\mathbb{I}_{\{\alpha_1^\varepsilon\geq\tau^1\}}]\\ &= \mathbb{E}_0[\mathbb{E}_0[\mathbb{I}_{\{\alpha_1^\varepsilon\geq\tau^1\}}|\mathcal{F}_{\tau^\varepsilon}]]\\ &= \mathbb{E}_0[\mathbb{P}_{Z(\tau^\varepsilon)}(\tau^0\geq\tau^1)],\end{aligned}\tag{4.8}$$

where recall that $\tau^0 = \inf\{t\geq 0: Z(t)\in H_0\}$. Similarly, once again conditioning on $\mathcal{F}_{\tau^\varepsilon}$ and using the strong Markov property of Z, we obtain

$$\begin{aligned}&\mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}]\\ &\quad= \mathbb{E}_0[\mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}|\mathcal{F}_{\tau^\varepsilon}]]\\ &\quad= \mathbb{E}_0[\mathbb{E}_{Z(\tau^\varepsilon)}[e^{-L(\tau^0\wedge\tau^1)}\mathbb{I}_{\{\tau^0<\tau^1\}}]].\end{aligned}$$

Therefore,

$$
\begin{aligned}
&1-\mathbb{E}_0[e^{-(L(\alpha_1^\varepsilon\wedge\tau^1)-L(\tau^\varepsilon\wedge\tau^1))}\mathbb{I}_{\{\alpha_1^\varepsilon<\tau^1\}}] \\
&\quad=\mathbb{E}_0[1-\mathbb{E}_{Z(\tau^\varepsilon)}[e^{-L(\tau^0\wedge\tau^1)}\mathbb{I}_{\{\tau^0<\tau^1\}}]] \\
&\quad=\mathbb{E}_0[\mathbb{P}_{Z(\tau^\varepsilon)}(\tau^0\geq\tau^1)]+\mathbb{E}_0[\mathbb{E}_{Z(\tau^\varepsilon)}[(1-e^{-L(\tau^0)})\mathbb{I}_{\{\tau^0<\tau^1\}}]].
\end{aligned}
\tag{4.9}
$$

The lemma follows from (4.7), (4.8) and (4.9). □

Next, we establish an elementary lemma that holds when the drift is zero. Recall the vector $\vec{\mathbf{v}}$ of property 2 of Definition 2.5.

LEMMA 4.4. *When $b\equiv 0$, the process $\langle Z,\vec{\mathbf{v}}\rangle$ is an $\mathcal{F}_t$-martingale on $[0,\tau^0]$ and for every $\varepsilon>0$, $\mathbb{P}_0$-a.s.,*

$$
\mathbb{P}_{Z(\tau^\varepsilon)}(\tau^0\geq\tau^1)=\varepsilon. \tag{4.10}
$$

PROOF. First, note that $H_0=\{0\}=\mathcal{V}$ by property 2 of Definition 2.5 and so $T_{\mathcal{V}}$ defined in (2.10) coincides with τ^0. From Lemma 2.8 and the continuity of the sample paths of Y, it follows that for $t\in[0,\tau^0]$, $\langle Y(t),\vec{\mathbf{v}}\rangle=0$ and so $\mathbb{P}$-a.s.,

$$
\langle Z(t),\vec{\mathbf{v}}\rangle=\langle Z(0),\vec{\mathbf{v}}\rangle+\tilde{M},\qquad t\in[0,\tau^0], \tag{4.11}
$$

where $\tilde{M}\doteq\langle\int_0^\cdot\sigma(Z(s))\cdot dB(s),\vec{\mathbf{v}}\rangle$ is an $\mathcal{F}_t$ martingale on $[0,\tau^0]$ since σ is uniformly bounded by property 3 of Definition 2.5. This establishes the first assertion of the lemma.

The quadratic variation $\langle\tilde{M}\rangle$ of $\tilde{M}$ is given by

$$
\langle\tilde{M}\rangle(t)=\int_0^t\vec{\mathbf{v}}^Ta(Z(s))\vec{\mathbf{v}}\,ds,\qquad t\in[0,\infty),
$$

where $a\doteq\sigma^T\sigma$. By property 4 of Definition 2.5, $a(\cdot)$ is uniformly elliptic. Therefore, $\mathbb{P}$-a.s., $\langle\tilde{M}\rangle$ is strictly increasing and $\langle\tilde{M}\rangle_\infty\doteq\lim_{t\to\infty}\langle\tilde{M}\rangle(t)=\infty$. For $t\in[0,\infty)$, let

$$
T(t)\doteq\inf\{s\geq 0:\langle\tilde{M}\rangle(s)>t\},\qquad\mathcal{G}_t\doteq\mathcal{F}_{T(t)},\qquad\tilde{B}(t)\doteq\tilde{M}(T(t)).
$$

Then $\{\tilde{B}_t,\mathcal{G}_t\}_{t\geq 0}$ is a standard one-dimensional Brownian motion (see, e.g., Theorem 4.6 on page 174 of [19]). Define $\tilde{\tau}^\varepsilon\doteq\inf\{t\geq 0:\tilde{B}(t)=\varepsilon\}$. By (4.11), we have $\mathbb{P}_0$-a.s.,

$$
\mathbb{P}_{Z(\tau^\varepsilon)}(\tau^0\geq\tau^1)=\mathbb{P}(\tilde{\tau}^0\geq\tilde{\tau}^1|\tilde{B}(0)=\varepsilon)=\varepsilon,
$$

where the latter follows from well-known properties of Brownian motion. This proves (4.10). □

REMARK 4.5. From Lemmas 4.3 and 4.4, we conclude that for every $\varepsilon > 0$,

$$\mathbb{E}[e^{-L(\tau^1)}] \leq \frac{\varepsilon}{\varepsilon + \mathbb{E}_0[\mathbb{E}_{Z(\tau^\varepsilon)}[(1 - e^{-L(\tau^\varepsilon)})\mathbb{I}_{\{\tau^0 < \tau^1\}}]]}.$$

Thus, in order to establish (4.3), it suffices to show that for some sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0$ as $k \to \infty$,

$$\liminf_{k\to\infty} \frac{1}{\varepsilon_k} \mathbb{E}_0[\mathbb{E}_{Z(\tau^{\varepsilon_k})}[(1 - e^{-L(\tau^{\varepsilon_k})})\mathbb{I}_{\{\tau^0 < \tau^1\}}]] = \infty.$$

This is established in Section 4.1.3 using scaling arguments. Since Z is a reflected diffusion (rather than just a reflected Brownian motion), the scaling arguments are more involved and rely on some weak convergence results that are established in Section 4.1.2. The reader may prefer to skip forward to the proof of Proposition 4.2 in Section 4.1.3 and refer back to the results in Section 4.1.2 when required.

4.1.2. *A weak convergence result.* Recall that we have assumed that the drift $b \equiv 0$. Now, let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and $\{x_k\}_{k\in\mathbb{N}}$ be sequences such that $\varepsilon_k \to 0$ as $k \to \infty$ and $x_k \in H_{\varepsilon_k}$ for $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $Z_{(k)}$ be the pathwise unique solution to the associated SDER with initial condition x_k, and let $X_{(k)}, Y_{(k)}$ and $L_{(k)}$ be the associated processes as defined in Definition 2.2 and (2.6). For $k \in \mathbb{N}$, consider the scaled process

$$B^k(t) \doteq \frac{B(\varepsilon_k^2 t)}{\varepsilon_k}, \qquad t \in [0, \infty),$$

which is a standard Brownian motion due to Brownian scaling. Similarly, define

$$A^k(t) \doteq \frac{A_{(k)}({\varepsilon_k}^2 t)}{\varepsilon_k}, \qquad A = X, Y, Z, L, \tag{4.12}$$

and let $\mathcal{F}^k_t \doteq \mathcal{F}_{{\varepsilon_k}^2 t}$ for $t \in [0, \infty)$. Clearly, the processes Z^k, B^k, Y^k and L^k are $\{\mathcal{F}^k_t\}$-adapted and $L^k(t) = \mathrm{Var}_{[0,t]} Y^k$ for every $t \geq 0$. For $(r, R) \in (0, \infty)^2$ such that $r < R$, let

$$\theta^k_{r,R} \doteq \inf\{t \geq 0 : \langle Z^k(t), \vec{\mathbf{v}}\rangle \notin (r, R)\}, \qquad k \in \mathbb{N}. \tag{4.13}$$

This section contains two main results. Roughly speaking, the first result (Lemma 4.7) shows that for the question under consideration, we can in effect replace the state-dependent diffusion coefficient $\sigma(\cdot)$ by $\sigma(0)$. This property is then used in Corollary 4.8 to provide bounds on the total variation sequence $L^k(\theta^k_{r,R})$, as $\varepsilon_k \to 0$. First, we observe that there exists a simple equivalence between (X^k, Z^k, Y^k) and another triplet of processes that will be easier to work with.

REMARK 4.6. For notational conciseness, we define the scaled diffusion coefficient

$$\sigma^k(x) \doteq \sigma(\varepsilon^k x), \qquad x \in \mathbb{R}^J, k \in \mathbb{N}.$$

By the definition of $Z_{(k)}$ and the scaling (4.12), it then follows that

$$X^k(t) = \frac{x_k}{\varepsilon_k} + \frac{1}{\varepsilon_k}\int_0^{\varepsilon_k^2 t} \sigma(Z_{(k)}(s))\,dB(s) = \frac{x_k}{\varepsilon_k} + \int_0^t \sigma^k(Z^k(s))\,dB^k(s),$$

where the last equality holds by the time-change theorem for stochastic integrals (see Proposition 1.4 in Chapter V of [25]). This implies Z^k is a strong solution to the SDER associated with $(G, d(\cdot))$, $b \equiv 0$, σ^k and the Brownian motion $\{B^k(t), \mathcal{F}_t^k\}_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with initial condition x_k/ε_k. If σ satisfies properties 3 and 4 of Definition 2.5 then so does σ^k, and thus $(G, d(\cdot))$, $b \equiv 0$ and σ^k also describe a Class $\mathcal{A}$ SDER. Therefore, by Theorem 2.7 there exists a pathwise unique solution $\tilde{Z}^k$ to the associated SDER for the Brownian motion $\{B_t, \mathcal{F}_t\}$ with initial condition x_k/ε_k. Let $\tilde{X}^k$ and $\tilde{Y}^k$ be the processes associated with $\tilde{Z}^k$, defined in the usual manner as follows:

$$\tilde{X}^k(t) = \frac{x_k}{\varepsilon_k} + \int_0^t \sigma^k(\tilde{Z}^k(s))\,dB(s), \qquad t \in [0,\infty), \tag{4.14}$$

and $\tilde{Y}^k = \tilde{Z}^k - \tilde{X}^k$. From the fact that solutions to Class $\mathcal{A}$ SDERs are unique in law by Theorem 2.7, it then follows that

$$(X^k, Z^k, Y^k) \stackrel{(d)}{=} (\tilde{X}^k, \tilde{Z}^k, \tilde{Y}^k), \tag{4.15}$$

where recall that $\stackrel{(d)}{=}$ indicates equality in distribution.

LEMMA 4.7. *Given $x_* \in \mathbb{R}_+^J$, let $(\overline{Z}, \overline{Y})$ satisfy the ESP pathwise for*

$$\overline{X} \doteq x_* + \sigma(0)B, \tag{4.16}$$

and let

$$\overline{\theta}_{r,R} \doteq \inf\{t \geq 0 : \langle \overline{Z}(t), \vec{\mathbf{v}}\rangle \notin (r,R)\}. \tag{4.17}$$

Suppose $b \equiv 0$ and $x_k/\varepsilon_k \to x_*$ as $k \to \infty$. Then the following properties hold:

(1) *As $k \to \infty$,*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\tilde{Z}^k(t) - \overline{Z}(t)|^2\Big] \to 0 \tag{4.18}$$

and $(X^k, Z^k, Y^k) \Rightarrow (\overline{X}, \overline{Z}, \overline{Y})$;

(2) *For all but countably many pairs* $(r,R)\in(0,\infty)^2$ *such that* $r<R$, *as* $k\to\infty$, *we have*

$$\max_{i=1,\dots,J}\sup_{s\in[0,\theta^k_{r,R}]} Y^k_i(s)\Rightarrow \max_{i=1,\dots,J}\sup_{s\in[0,\overline{\theta}_{r,R}]}\overline{Y}_i(s).$$

PROOF. Note that since $x_k/\varepsilon_k\in H_1$ for every $k\in\mathbb{N}$ and H_1 is closed, we must have $x_*\in H_1$. We first prove property 1. Let $\tilde{X}^k,\tilde{Z}^k$ and $\tilde{Y}^k$ be as in Remark 4.6. Then, by (4.15), it clearly suffices to show that $(\tilde{X}^k,\tilde{Z}^k,\tilde{Y}^k)\Rightarrow(\overline{X},\overline{Z},\overline{Y})$. From (4.14) and (4.16), it follows that for $t\in[0,\infty)$,

$$\begin{aligned}
|\tilde{X}^k(t)-\overline{X}(t)|^2 &\le \left|\frac{x_k}{\varepsilon_k}-x_*+\int_0^t(\sigma^k(\tilde{Z}^k(s))-\sigma(0))\,dB(s)\right|^2\\
&\le\left(\left|\frac{x_k}{\varepsilon_k}-x_*\right|+\left|\int_0^t(\sigma^k(\overline{Z}(s))-\sigma(0))\,dB(s)\right|\right.\\
&\qquad\left.+\left|\int_0^t\Big(\sigma^k(\tilde{Z}^k(s))-\sigma^k(\overline{Z}(s))\Big)\,dB(s)\right|\right)^2.
\end{aligned}$$

Using the fact that $(a+b+c)^2\le 3(a^2+b^2+c^2)$ for all $a,b,c\in\mathbb{R}$ and taking the supremum over $t\in[0,T]$ and then expectations of both sides, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}|\tilde{X}^k(t)-\overline{X}(t)|^2\Big]\\
&\quad\le 3\left|\frac{x_k}{\varepsilon_k}-x_*\right|^2+3\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t(\sigma^k(\overline{Z}(s))-\sigma(0))\,dB(s)\right|^2\right]\\
&\qquad+3\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t(\sigma^k(\tilde{Z}^k(s))-\sigma^k(\overline{Z}(s)))\,dB(s)\right|^2\right].
\end{aligned}$$

Since σ is uniformly bounded, the stochastic integrals on the right-hand side are martingales. By applying the Burkholder–Davis–Gundy (BDG) inequality, the Lipschitz condition on σ, the definition of σ^k and Fubini's theorem, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t(\sigma^k(\tilde{Z}^k(s))-\sigma^k(\overline{Z}(s)))\,dB(s)\right|^2\right]\\
&\quad\le C_2\mathbb{E}\left[\int_0^T|\sigma^k(\tilde{Z}^k(s))-\sigma^k(\overline{Z}(s))|^2\,ds\right]\\
&\quad\le C_2\tilde{K}^2\varepsilon_k^2\mathbb{E}\left[\int_0^T|\tilde{Z}^k(s)-\overline{Z}(s)|^2\,ds\right]\\
&\quad\le C_2\tilde{K}^2\varepsilon_k^2\int_0^T\mathbb{E}\Big[\sup_{u\in[0,s]}|\tilde{Z}^k(u)-\overline{Z}(u)|^2\Big]\,ds,
\end{aligned}$$

where $C_2 < \infty$ is the universal constant in the BDG inequality. Using similar arguments, we also see that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t (\sigma^k(\overline{Z}(s)) - \sigma(0))\,dB(s)\Big|^2\Big] \le C_2\tilde{K}^2\varepsilon_k^2\int_0^T \mathbb{E}\Big[\sup_{u\in[0,s]}|\overline{Z}(u)|^2\Big]\,ds$$
$$\le C_2\tilde{K}^2\varepsilon_k^2 T\mathbb{E}\Big[\sup_{t\in[0,T]}|\overline{Z}(t)|^2\Big].$$

Combining the last three displays, and setting $\tilde{C}_T \doteq 3C_2\tilde{K}^2(1\vee T) < \infty$, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\tilde{X}^k(t) - \overline{X}(t)|^2\Big] \le \tilde{C}_T\varepsilon_k^2\int_0^T \mathbb{E}\Big[\sup_{u\in[0,s]}|\tilde{Z}^k(u) - \overline{Z}(u)|^2\Big]\,ds + R^k(T), \tag{4.19}$$

where

$$R^k(T) \doteq 3\Big|\frac{x_k}{\varepsilon_k} - x_*\Big|^2 + \tilde{C}_T\varepsilon_k^2\mathbb{E}\Big[\sup_{t\in[0,T]}|\overline{Z}(t)|^2\Big].$$

By the assumed Lipschitz continuity of $\overline{\Gamma}$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\overline{Z}(t)|^2\Big] \le K_T^2\mathbb{E}\Big[\sup_{t\in[0,T]}|x_* + \sigma(0)B(t)|^2\Big]$$
$$\le 2K_T^2|x_*|^2 + 2K_T^2|\sigma(0)|^2\mathbb{E}\Big[\sup_{t\in[0,T]}|B(t)|^2\Big] < \infty.$$

Since $x_k/\varepsilon_k \to x_*$ and $\varepsilon_k \to 0$ as $k\to\infty$, it follows that

$$\lim_{k\to\infty} R^k(T) = 0. \tag{4.20}$$

On the other hand, combining the inequality in (4.19) with the Lipschitz continuity of the map $\bar{\Gamma}$, we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\tilde{Z}^k(t) - \overline{Z}(t)|^2\Big]$$
$$\le K_T^2 R^k(T) + K_T^2\tilde{C}_T\varepsilon_k^2\int_0^T \mathbb{E}\Big[\sup_{u\in[0,s]}|\tilde{Z}^k(u) - \overline{Z}(u)|^2\Big]\,ds.$$

An application of Gronwall's lemma then shows that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\tilde{Z}^k(t) - \overline{Z}(t)|^2\Big] \le K_T^2 R^k(T)e^{K_T^2\tilde{C}_T\varepsilon_k^2},$$

which converges to zero as $k \to \infty$ due to (4.20) and the fact that $\varepsilon_k \to 0$ as $k \to \infty$. This proves (4.18). In turn, substituting the last inequality back into (4.19) and, again using (4.20) and the fact that $\varepsilon_k \to 0$, we also obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\tilde{X}^k(t) - \overline{X}(t)|^2\Big] \to 0 \qquad \text{as } k \to \infty,$$

which implies $\tilde{X}^k \Rightarrow \overline{X}$. Since the mapping from $\tilde{X}^k \mapsto (\tilde{X}^k, \tilde{Z}^k, \tilde{Y}^k)$ is continuous, by the continuous mapping theorem it follows that $(\tilde{X}^k, \tilde{Z}^k, \tilde{Y}^k) \Rightarrow (\overline{X}, \overline{Z}, \overline{Y})$ and the first property of the lemma is established.

We now turn to the proof of the second property. By the first property, we know that $(Z^k, Y^k) \Rightarrow (\overline{Z}, \overline{Y})$ as $k \to \infty$. This immediately implies that for all but countably main pairs $(r, R) \in (0, \infty)^2$ such that $r < R$, we have, as $k \to \infty$,

$$(Z^k(\cdot \wedge \theta^k_{r,R}), Y^k(\cdot \wedge \theta^k_{r,R}), \theta^k_{r,R}) \Rightarrow (\overline{Z}(\cdot \wedge \overline{\theta}_{r,R}), \overline{Y}(\cdot \wedge \overline{\theta}_{r,R}), \overline{\theta}_{r,R}).$$

(For an argument that justifies this implication, see, e.g., the proof of Theorem 4.1 on page 354 of [11].) Using the continuity of the map $(f, g, t) \mapsto \max_{i=1,\dots,J} \sup_{s\in[0,t]} g_i(s)$ from $\mathcal{C}[0,\infty) \times \mathcal{C}[0,\infty) \times \mathbb{R}_+$ to $\mathbb{R}_+$, an application of the continuous mapping theorem yields the second property. $\square$

COROLLARY 4.8. *Suppose $b \equiv 0$ and $x_k/\varepsilon_k \to x_*$ as $k \to \infty$. Then for each pair $(r, R) \in (0, \infty)$ such that $r < R$, the following properties hold:*

(1) $\mathbb{P}(\sup_{k\in\mathbb{N}} L^k(\theta^k_{r,R}) < \infty) = 1$.
(2) $\varepsilon_k L^k(\theta^k_{r,R}) \Rightarrow 0$.
(3) $\mathbb{P}(\overline{L}(\overline{\theta}_{r,R}) < \infty) = 1$ *and if* $r < \langle x, \vec{\mathbf{v}}\rangle < R$, $\mathbb{P}(\overline{L}(\overline{\theta}_{r,R}) > 0) > 0$.

PROOF. If $\langle x_*, \vec{\mathbf{v}}\rangle < r$ or $\langle x_*, \vec{\mathbf{v}}\rangle > R$, then $\overline{\theta}_{r,R} = 0$ and $\theta^k_{r,R} = 0$ for all k sufficiently large. In this case, properties (1)–(3) hold trivially. Hence, for the rest of the proof, we assume that $r \le \langle x_*, \vec{\mathbf{v}}\rangle \le R$.

We start by proving property 1. Let $\tilde{X}^k$, $\tilde{Z}^k$ and $\tilde{Y}^k$ be defined as in Remark 4.6, and let $\tilde{L}^k$ be defined as in (2.6), but with Y replaced by $\tilde{Y}^k$. By (4.15), it follows that $(L^k, \theta^k_{r,R})$ and $(\tilde{L}^k, \tilde{\theta}^k_{r,R})$ have the same distribution for each $k \in \mathbb{N}$, where $\tilde{\theta}^k_{r,R}$ is defined in the obvious way:

$$\tilde{\theta}^k_{r,R} \doteq \inf\{t \ge 0 : \langle \tilde{Z}^k(t), \vec{\mathbf{v}}\rangle \notin (r, R)\}.$$

We now argue that $\mathbb{P}(\tilde{\theta}^k_{r,R} < \infty) = 1$. Indeed, for $k \in \mathbb{N}$ such that $x_k/\varepsilon_k \notin (r, R)$ this holds trivially. On the other hand, if $\tilde{Z}^k(0) = x_k/\varepsilon_k \in (r, R)$ then this follows because Lemma 2.8 and the uniform ellipticity condition show that, on $(0, \tilde{\theta}_{r,R})$, $\langle \tilde{Z}^k(t), \vec{\mathbf{v}}\rangle = \langle \tilde{X}^k(t), \vec{\mathbf{v}}\rangle$ is a continuous martingale whose

quadratic variation is strictly bounded away from zero. Thus, $\langle \tilde{Z}^k, \vec{\mathbf{v}}\rangle$ is $\mathbb{P}$-a.s. unbounded, and hence $\tilde{\theta}^k_{r,R}$ is $\mathbb{P}$-a.s. finite. Therefore, to prove property 1, it suffices to show that

$$\mathbb{P}\Big(\sup_{k\in\mathbb{N}} \tilde{L}^k(\tilde{\theta}^k_{r,R} \wedge T) < \infty\Big) = 1, \qquad T > 0.$$

Fix $T \in (0,\infty)$. Since $r > 0$, there exists $\delta > 0$ such that $\langle y, \vec{\mathbf{v}}\rangle < r$ for all y with $|y| \le \delta$. Let $\tilde{\kappa}^k_\delta \doteq \inf\{t \ge 0 : |\tilde{Z}^k(t)| \le \delta\}$. Then $\theta^k_{r,R} \le \tilde{\kappa}^k_\delta$ for all $k \in \mathbb{N}$. Let

$$\tilde{C}^k \doteq \sup_{t\in[0,T]} |\tilde{Z}^k(t)| \vee |\tilde{X}^k(t)|.$$

By property 1 of Lemma 4.7, it follows that $(\tilde{X}^k, \tilde{Z}^k) \Rightarrow (\overline{X}, \overline{Z})$ as $k \to \infty$. Using the continuity of the map $(f,g) \mapsto \sup_{s\in[0,T]} |f(s)| \vee |g(s)|$ from $\mathcal{C}[0,\infty) \times \mathcal{C}[0,\infty)$ to $\mathbb{R}_+$, an application of the continuous mapping theorem yields $\tilde{C}^k \Rightarrow \overline{C}$ as $k \to \infty$, where $\overline{C} \doteq \sup_{t\in[0,T]} |\overline{Z}(t)| \vee |\overline{X}(t)|$. Also, due to the Lipschitz continuity of the ESM $\bar{\Gamma}$ and (4.16), $\mathbb{P}$-a.s., we have

$$\sup_{t\in[0,T]} |\overline{Z}(t)| \le K_T \sup_{t\in[0,T]} |\overline{X}(t)| \le K_T\Big(|x_*| + |\sigma(0)| \sup_{s\in[0,T]} |B(s)|\Big) < \infty,$$

and hence $\mathbb{P}$-a.s., $\overline{C} < \infty$. It then follows that $\mathbb{P}(\sup_{k\in\mathbb{N}} \tilde{C}^k < \infty) = 1$. Moreover, $\mathcal{V} = \{0\}$ and for each $\omega \in \Omega$, $(\overline{Z}(\cdot,\omega), \overline{Y}(\cdot,\omega))$ solves the ESP for $\overline{X}(\cdot,\omega)$. Therefore, it follows from Lemma 2.8 of [22] that there exist $\rho > 0$, independent of k, a finite set $\mathbb{I} = \{1,\dots,I\}$ and a collection of open sets $\{\mathcal{O}_i, i \in \mathbb{I}\}$ of $\mathbb{R}^J$ and associated vectors $\{v_i \in S_1(0), i \in \mathbb{I}\}$ that satisfy the following two properties:

(1) $[\{x \in G : |x| \le \overline{C}\} \setminus N_{\delta/2}(0)^\circ] \subset [\bigcup_{i\in\mathbb{I}} \mathcal{O}_i]$.
(2) If $y \in \{x \in G : |x| \le \overline{C}\} \cap N_\rho(\mathcal{O}_i)$ for some $i \in \mathbb{I}$ then $\langle d, v_i\rangle \ge \rho$ for every $d \in d(y)$ with $|d| = 1$.

Moreover, as in the proof of Theorem 2.9 of [22], for each $\omega \in \Omega$, we can define a sequence $\{(\overline{T}_m(\omega), \bar{i}_m(\omega)), m = 0,1,\dots\}$ defined recursively as follows. Let $\overline{T}_0(\omega) \doteq 0$ and let $\bar{i}_0(\omega) \in \mathbb{I}$ be such that $\overline{Z}(0,\omega) = x_* \in \mathcal{O}_{\bar{i}_0(\omega)}$. Note that because $x_* \in (r,R)$ implies $|x| > \delta > \delta/2$, such an i_0 exists by property (1) above. Next, for each $m = 0,1,\dots,$ whenever $\overline{T}_m(\omega) < \overline{\kappa}_{\delta/2}(\omega) \doteq \inf\{t \ge 0 : \overline{Z}(t,\omega) \in N_{\delta/2}(0)\}$, define

$$\overline{T}_{m+1}(\omega) \doteq \inf\{t > \overline{T}_m(\omega) : \overline{Z}(t,\omega) \notin N_{\rho/2}(\mathcal{O}_{\bar{i}_m(\omega)})^\circ \text{ or } \overline{Z}(t,\omega) \in N_{\delta/2}(0)\}.$$

If $\overline{T}_{m+1}(\omega) < T \wedge \overline{\kappa}_{\delta/2}(\omega)$, choose $\bar{i}_{m+1}(\omega) \in \mathbb{I}$ such that $\overline{Z}(\overline{T}_{m+1}(\omega),\omega) \in \mathcal{O}_{\bar{i}_{m+1}(\omega)}$. Note that such an $i_{m+1}(\omega)$ exists by property (1) above. Let $\overline{N}(\omega) < \infty$ be the smallest integer such that $\overline{T}_{\overline{N}(\omega)}(\omega) \ge T \wedge \overline{\kappa}_{\delta/2}(\omega)$ and redefine $\overline{T}_{\overline{N}(\omega)}(\omega) = T \wedge \overline{\kappa}_{\delta/2}(\omega)$. (Note that $\overline{N}(\omega)$ and $\{(\overline{T}_m(\omega), \bar{i}_m(\omega)), m =$

$0,1,\dots\}$ are constructed in the same way as M and $\{T_m, m\in\mathbb{N}\}$ in Theorem 2.9 of [22], except that we replace ρ and δ by $\rho/2$ and $\delta/2$, respectively.)

Since, as shown in Lemma 4.7, $(X^k, Z^k, Y^k) \Rightarrow (\overline{X}, \overline{Z}, \overline{Y})$ as $k\to\infty$ and $(\overline{X}, \overline{Z}, \overline{Y})$ has continuous paths, by invoking the Skorokhod representation theorem, we may assume without loss of generality that there exists $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that for every $\omega \in \tilde{\Omega}$, $(X^k(\omega), Z^k(\omega), Y^k(\omega)) \to (\overline{X}(\omega), \overline{Z}(\omega), \overline{Y}(\omega))$ uniformly on $[0,T]$ as $k\to\infty$. Let $\bar{k} < \infty$ be such that for all $k > \bar{k}$, $\sup_{t\in[0,T]} |Z^k(t,\omega) - \overline{Z}(t,\omega)| < (\rho\wedge\delta)/4$. Then $Z^k(\cdot,\omega)$ will stay in $N_\rho(\mathcal{O}_{\bar{i}_m(\omega)})$ during the interval $[\overline{T}_m(\omega), \overline{T}_{m+1}(\omega))$. Exactly as in the proof of Lemma 2.9 of [22] (note that the argument there only requires that $\phi(t) \in N_\rho(\mathcal{O}_{k_{m-1}})$ for $t\in[T_{m-1}, T_m)$), we can then argue that $\tilde{L}^k(T\wedge\tau^k_\delta(\omega), \omega) \le (4\tilde{C}^k(\omega)\overline{N}(\omega))/\rho$ for $\omega\in\tilde{\Omega}$. Together with the fact that $\mathbb{P}(\sup_{k\in\mathbb{N}} \tilde{C}^k < \infty) = 1$ and $\overline{N}(\omega) < \infty$ for each $\omega\in\Omega$, this shows that $\mathbb{P}(\tilde{L}^k(\tilde{\tau}^k_\delta \wedge T) < \infty) = 1$. Since $\tilde{L}^k(\tilde{\theta}^k_{r,R}\wedge T) \le \tilde{L}^k(\tilde{\tau}^k_\delta\wedge T)$, we then have $\mathbb{P}(\sup_{k\in\mathbb{N}} \tilde{L}^k(\tilde{\theta}^k_{r,R}\wedge T) < \infty) = 1$. This completes the proof of property 1.

Property 2 follows directly from property 1 and the fact that $\varepsilon_k \to 0$ as $k\to\infty$. In addition, by Theorem 2.7 it follows that $\overline{Z}$ is a semimartingale on $[0, T_{\mathcal{V}})$, with $\overline{Y}$ being the bounded variation term in the decomposition. The first assertion of property 3 is thus a direct consequence of the fact that $\overline{\theta}_{r,R} < T_{\mathcal{V}}$. For the second assertion of property 3, notice that with positive probability, the Brownian motion $\overline{X} = x_* + \sigma(0)B$ will exit G before it hits one of the two levels H_r or H_R. Since $\overline{Z}$ lies in G and $\overline{Z} = \overline{X} + \overline{Y}$, this implies that, with positive probability, $\overline{Y}$ is not identically zero in the interval $[0, \overline{\theta}_{r,R})$. This, in turn, implies that $\overline{L}(\overline{\theta}_{r,R})$ is strictly positive with positive probability. Thus, the second assertion of property 3 is also established, and the proof of the corollary is complete. $\square$

4.1.3. *A scaling argument.* Since the equality $\mathbb{E}_0[e^{-L(\tau^1)}] = 0$ implies that $\mathbb{P}$-a.s., $L(\tau^1) = \infty$, in order to prove Proposition 4.2 it suffices to establish the former equality. In turn, by Remark 4.5, this equality holds if there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0$ as $k\to\infty$, and

$$\liminf_{k\to\infty} \frac{1}{\varepsilon_k} \mathbb{E}_0[\mathbb{E}_{Z(\tau^{\varepsilon_k})}[(1 - e^{-L(\tau^0)})\mathbb{I}_{\{\tau^0 < \tau^1\}}]] = \infty. \tag{4.21}$$

We will show that (4.21) holds by using the strong Markov property and scaling arguments. First, we need to introduce some additional notation. Fix $\varepsilon > 0$. Let Λ_ε denote the following union of hyperplanes:

$$\Lambda_\varepsilon \doteq \bigcup_{n\in\mathbb{Z}} H_{2^n\varepsilon}. \tag{4.22}$$

For $x\in\Lambda_\varepsilon$, let $N_\varepsilon(x)$ denote the pair of hyperplanes in Λ_ε that are adjacent to the hyperplane on which x lies. In other words, let

$$N_\varepsilon(x) \doteq H_{2^{n-1}\varepsilon} \cup H_{2^{n+1}\varepsilon}, \qquad x\in H_{2^n\varepsilon}, n\in\mathbb{Z}. \tag{4.23}$$

For future reference, note that for $y \in \mathbb{R}_+^J$ and $x \in H_{2^n\varepsilon}$, $n \in \mathbb{Z}$,

$$\frac{y}{\varepsilon} \in N_1\left(\frac{x}{\varepsilon}\right) \quad \Rightarrow \quad y \in N_\varepsilon(x). \tag{4.24}$$

Let $\{\beta_n^\varepsilon\}_{n\in\mathbb{N}}$ be the sequence of random times defined recursively by $\beta_0^\varepsilon \doteq 0$ and for $n \in \mathbb{N}$,

$$\beta_n^\varepsilon \doteq \inf\{t \geq \beta_{n-1}^\varepsilon : Z(t) \in N_\varepsilon(Z(\beta_{n-1}^\varepsilon))\}. \tag{4.25}$$

It is easy to see that $\{\beta_n^\varepsilon\}_{n\in\mathbb{N}}$ defines a sequence of stopping times (for completeness, a proof is provided in Lemma B.1).

Observe that L is nondecreasing and for $x \in H_\varepsilon$, $\mathbb{P}_x$-a.s., $\beta_n^\varepsilon \leq \tau_0$ for every $n \in \mathbb{N}$. Now $Z(\tau^\varepsilon) \neq 0$ because $\varepsilon > 0$. Hence, for every $n \in \mathbb{N}$,

$$\mathbb{E}_{Z(\tau^\varepsilon)}[(1 - e^{-L(\tau^0)})\mathbb{I}_{\{\tau^0<\tau^1\}}] \geq \mathbb{E}_{Z(\tau^\varepsilon)}[(1 - e^{-L(\beta_n^\varepsilon)})\mathbb{I}_{\{\tau^0<\tau^1\}}]. \tag{4.26}$$

Using the elementary identity

$$1 - e^{-L(\beta_n^\varepsilon)} = 1 - e^{-L(\beta_{n-1}^\varepsilon)} + e^{-L(\beta_{n-1}^\varepsilon)}(1 - e^{-(L(\beta_n^\varepsilon)-L(\beta_{n-1}^\varepsilon))}),$$

conditioning on $\mathcal{F}_{\beta_{n-1}^\varepsilon}$, and invoking the strong Markov property of Z, the right-hand side of (4.26) can be expanded as

$$\begin{aligned}
&\mathbb{E}_{Z(\tau^\varepsilon)}[(1 - e^{-L(\beta_n^\varepsilon)})\mathbb{I}_{\{\tau^0<\tau^1\}}] \\
&\quad = \mathbb{E}_{Z(\tau^\varepsilon)}[(1 - e^{-L(\beta_{n-1}^\varepsilon)})\mathbb{I}_{\{\tau^0<\tau^1\}}] \\
&\qquad + \mathbb{E}_{Z(\tau^\varepsilon)}[\mathbb{E}_{Z(\tau^\varepsilon)}[e^{-L(\beta_{n-1}^\varepsilon)}(1 - e^{-(L(\beta_n^\varepsilon)-L(\beta_{n-1}^\varepsilon))})\mathbb{I}_{\{\tau^0<\tau^1\}}|\mathcal{F}_{\beta_{n-1}^\varepsilon}]] \\
&\quad = \mathbb{E}_{Z(\tau^\varepsilon)}[(1 - e^{-L(\beta_{n-1}^\varepsilon)})\mathbb{I}_{\{\tau^0<\tau^1\}}] \\
&\qquad + \mathbb{E}_{Z(\tau^\varepsilon)}[e^{-L(\beta_{n-1}^\varepsilon)}\mathbb{E}_{Z(\beta_{n-1}^\varepsilon)}[(1 - e^{-L(\beta_1^\varepsilon)})\mathbb{I}_{\{\tau^0<\tau^1\}}]].
\end{aligned}$$

Observing that the first term on the right-hand side is identical to the term on the left-hand side, except for a shift down in the index n, we can iterate this procedure and use the relation $L(\beta_0^\varepsilon) = L(0) = 0$ to conclude that for any $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}_{Z(\tau^\varepsilon)}[(1 - e^{-L(\beta_n^\varepsilon)})\mathbb{I}_{\{\tau^0<\tau^1\}}] \\
&\quad = \sum_{m=1}^{n} \mathbb{E}_{Z(\tau^\varepsilon)}[e^{-L(\beta_{m-1}^\varepsilon)}\mathbb{E}_{Z(\beta_{m-1}^\varepsilon)}[(1 - e^{-L(\beta_1^\varepsilon)})\mathbb{I}_{\{\tau^0<\tau^1\}}]].
\end{aligned} \tag{4.27}$$

Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and $\{x_k\}_{k\in\mathbb{N}}$ be sequences such that $x_k \in H_{\varepsilon_k}$ for $k \in \mathbb{N}$, and $\varepsilon_k \to 0$ as $k \to \infty$. Since H_1 is compact and $x_k/\varepsilon_k \in H_1$ for every $k \in \mathbb{N}$, we can assume without loss of generality (by choosing an appropriate subsequence, if necessary) that there exists $x_* \in H_1$ such that $x_k/\varepsilon_k \to x_*$, as $k \to \infty$. We now show that, when ε is replaced by ε_k, each term in the

sum on the right-hand side of (4.27), is $O(\varepsilon_k)$ (as $k\to\infty$), with a constant that is independent of m. This proof relies on the estimates obtained in the next two lemmas. In both lemmas, $Z_{(k)}, Y_{(k)}, L_{(k)}$, Z^k, Y^k and L^k denote the processes defined at the beginning of Section 4.1.2, and for $\varepsilon>0$, let $\beta^{\varepsilon}_{(k),0}\doteq 0$ and for $n\in\mathbb{N}$,

$$\beta^{\varepsilon}_{(k),n}\doteq\inf\{t\geq\beta^{\varepsilon}_{(k),n-1}: Z_{(k)}(t)\in N_\varepsilon(Z_{(k)}(\beta^{\varepsilon}_{(k),n-1}))\},\tag{4.28}$$

and, likewise, let $\zeta_{k,0}\doteq 0$ and for $n\in\mathbb{N}$, define

$$\zeta_{k,n}\doteq\inf\{t\geq\zeta_{k,n-1}: Z^k(t)\in N_1(Z^k(\zeta_{k,n-1}))\}.\tag{4.29}$$

Note that these sequences of stopping times are defined in a manner analogous to the sequence $\{\beta^\varepsilon_n\}_{n\in\mathbb{N}}$ defined in (4.25), except that Z is replaced by $Z_{(k)}$ and Z^k, respectively. Moreover, these definitions, together with the scaling relations (4.12) and (4.24), yield the following equivalence relation

$$\varepsilon_k^2\zeta_{k,n}=\beta^{\varepsilon_k}_{(k),n},\qquad k,n\in\mathbb{N},\tag{4.30}$$

LEMMA 4.9. *Suppose $b\equiv 0$. Then there exists $C>0$ such that*

$$\liminf_{k\to\infty}\frac{1}{\varepsilon_k}\inf_{x\in H_{\varepsilon_k}}\mathbb{E}_x[(1-e^{-L(\beta_1^{\varepsilon_k})})\mathbb{I}_{\{\tau^0<\tau^1\}}]\geq C.\tag{4.31}$$

PROOF. Since the law of $(Z_{(k)}, Y_{(k)}, L_{(k)})$ under $\mathbb{P}$ is the same as the law of (Z,Y,L) under $\mathbb{P}_{x_k}$, we have

$$\begin{aligned}&\liminf_{k\to\infty}\frac{1}{\varepsilon_k}\mathbb{E}_{x_k}[(1-e^{-L(\beta_1^{\varepsilon_k})})\mathbb{I}_{\{\tau^0<\tau^1\}}]\\&\quad=\liminf_{k\to\infty}\frac{1}{\varepsilon_k}\mathbb{E}[(1-e^{-L_{(k)}(\beta^{\varepsilon_k}_{(k),1})})\mathbb{I}_{\{\tau^0_{(k)}<\tau^1_{(k)}\}}],\end{aligned}\tag{4.32}$$

where $\tau^\varepsilon_{(k)}$ and $\tau^{k,\varepsilon}$ are defined as follows:

$$\begin{aligned}\tau^\varepsilon_{(k)}&\doteq\inf\{t\geq 0: Z_{(k)}(t)\in H_\varepsilon\},\\ \tau^{k,\varepsilon}&\doteq\inf\{t\geq 0: Z^k(t)\in H_{\varepsilon/\varepsilon_k}\},\end{aligned}\tag{4.33}$$

and recall the definition of $\beta^\varepsilon_{(k),1}$ given in (4.28). Assume, without loss of generality, that k is large enough so that $\varepsilon_k<1$. Then, for each $x\geq 0$, applying the mean value theorem to the function $f_x(\varepsilon)=1-e^{-\varepsilon x}$, we infer that for $x\geq 0$, there exists $\varepsilon_k^*=\varepsilon_k^*(x)\in(0,\varepsilon_k)$ such that

$$\frac{1-e^{-\varepsilon_k x}}{\varepsilon_k}=xe^{-\varepsilon_k^* x}\geq xe^{-x}.$$

Using the above inequality along with the equalities $L_{(k)}(\beta^{\varepsilon_k}_{(k),1}) = \varepsilon_k L^k(\zeta_{k,1})$, $\varepsilon_k^2 \tau^{k,0} = \tau^0_{(k)}$ and $\varepsilon_k^2 \tau^{k,1} = \tau^1_{(k)}$, which hold due to the scaling relations (4.12) and (4.24), we have for all k sufficiently large,

$$\frac{1}{\varepsilon_k}\mathbb{E}[(1 - e^{-L_{(k)}(\beta^{\varepsilon_k}_{(k),1})})\mathbb{I}_{\{\tau^0_{(k)} < \tau^1_{(k)}\}}] = \mathbb{E}\left[\left(\frac{1 - e^{-\varepsilon_k L^k(\zeta_{k,1})}}{\varepsilon_k}\right)\mathbb{I}_{\{\tau^{k,0} < \tau^{k,1}\}}\right]$$

$$\geq \mathbb{E}[L^k(\beta^{k,1}_1) e^{-L^k(\beta^{k,1}_1)}\mathbb{I}_{\{\tau^{k,0} < \tau^{k,1}\}}].$$

Comparing this with (4.31) and (4.32), it is clear that to prove the lemma it suffices to show that there exists $\tilde{C} > 0$ such that

$$\liminf_{k\to\infty} \mathbb{E}[L^k(\zeta_{k,1}) e^{-L^k(\zeta_{k,1})}\mathbb{I}_{\{\tau^{k,0} < \tau^{k,1}\}}] \geq \tilde{C}. \tag{4.34}$$

Let $\overline{X} = x_* + \sigma(0)B$, where B is standard Brownian motion, and let $(\overline{Z}, \overline{Y})$ satisfy the ESP for $\overline{X}$, as in Lemma 4.7. Then, since $x_k/\varepsilon_k \to x_* \in H_1$, by Lemma 4.7(2) it follows that there exist $r \in (1/2, 1)$ and $R \in (1, 2)$ such that as $k \to \infty$,

$$\max_{i=1,\dots,J} \sup_{s\in[0,\theta^k_{r,R}]} Y^k_i(s) \Rightarrow \max_{i=1,\dots,J} \sup_{s\in[0,\overline{\theta}_{r,R}]} \overline{Y}_i(s),$$

where recall the definitions of $\theta^k_{r,R}$ and $\overline{\theta}_{r,R}$ given in (4.13) and (4.17), respectively. By the Portmanteau theorem, this implies that

$$\mathbb{P}\Big(\max_{i=1,\dots,J} \sup_{t\in[0,\overline{\theta}_{r,R}]} \overline{Y}_i(t) > \delta\Big) \leq \liminf_{k\to\infty} \mathbb{P}\Big(\max_{i=1,\dots,J} \sup_{t\in[0,\theta^k_{r,R}]} Y^k_i(t) > \delta\Big)$$

$$\leq \liminf_{k\to\infty} \mathbb{P}(L^k(\theta^k_{r,R}) > \delta).$$

Together with the fact that property 3 of Corollary 4.8 implies that there exists $\delta > 0$ such that

$$\mathbb{P}\Big(\max_{i=1,\dots,J} \sup_{t\in[0,\overline{\theta}_{r,R}]} \overline{Y}_i(t) > \delta\Big) > 2\delta,$$

and the inequality $\zeta_{k,1} \geq \theta^k_{r,R}$ for all k, it follows that there exists $K < \infty$ such that

$$\mathbb{P}(L^k(\zeta_{k,1}) > \delta) \geq \delta, \qquad k \geq K. \tag{4.35}$$

Next, choose $r' \in (0, 1/2)$ and $R' \in (2, \infty)$ and note that $\zeta_{k,1} \leq \theta^k_{r',R'}$ because $Z^k(0) \in H_1$ and $N_1(Z^k(0)) = H_{1/2} \cup H_2$. Hence, property 1 of Corollary 4.8 implies that there exists $c < \infty$ such that

$$\sup_{k\in\mathbb{N}} \mathbb{P}(L^k(\theta^k_{r',R'}) < c) \geq \mathbb{P}\Big(\sup_{k\in\mathbb{N}} L^k(\theta^k_{r',R'}) < c\Big) \geq 1 - \frac{\delta}{4}. \tag{4.36}$$

On the other hand, since $\mathbb{P}_{Z(\tau^{\varepsilon_k})}(\tau^0 \geq \tau^1) = \varepsilon_k$ by Lemma 4.4 and $\varepsilon_k \to 0$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \mathbb{P}(\tau^{k,0} < \tau^{k,1}) = \lim_{k\to\infty} \mathbb{P}(\tau^0_{(k)} < \tau^1_{(k)}) = \lim_{k\to\infty} (1-\varepsilon_k) = 1.$$

Hence, by choosing $K < \infty$ larger if necessary, we can assume that

$$\mathbb{P}(\tau^{k,0} < \tau^{k,1}) \geq 1 - \frac{\delta}{4}, \qquad k \geq K. \tag{4.37}$$

Now, define the set

$$S_k \doteq \{\tau^{k,0} < \tau^{k,1}, e^{-L^k(\zeta_{k,1})} \geq e^{-c}, L^k(\zeta_{k,1}) > \delta\}.$$

Then (4.35), (4.36) and (4.37), together show that for $k \geq K$, $\mathbb{P}(S_k) \geq \delta/2$. Therefore, for all $k \geq K$,

$$\begin{aligned}
&\mathbb{E}[L^k(\zeta_{k,1}) e^{-L^k(\zeta_{k,1})} \mathbb{I}_{\{\tau^{k,0}<\tau^{k,1}\}}] \\
&\qquad \geq \mathbb{E}[L^k(\zeta_{k,1}) e^{-L^k(\zeta_{k,1})} \mathbb{I}_{\{\tau^{k,0}<\tau^{k,1}\}} \mathbb{I}_{S_k}] \geq \delta e^{-c} \frac{\delta}{2},
\end{aligned}$$

and so (4.34) holds with $\tilde{C} = \delta^2 e^{-c}/2$. This completes the proof of the lemma. □

LEMMA 4.10. *Suppose $b \equiv 0$. For every $n \in \mathbb{N}$,*

$$\lim_{k\to\infty} \sup_{x\in H_{\varepsilon_k}} \mathbb{E}_x[1 - e^{-L(\beta_n^{\varepsilon_k})}] = 0.$$

PROOF. Fix $n \in \mathbb{N}$. We prove the lemma using an argument by contradiction. Suppose that there exists $\delta_0 > 0$ and a subsequence, which we denote again by $\{\varepsilon_k\}_{k\in\mathbb{N}}$, such that $\varepsilon_k \downarrow 0$ as $k \to \infty$ and for every $k \in \mathbb{N}$,

$$\sup_{x\in H_{\varepsilon_k}} \mathbb{E}_x[1 - e^{-L(\beta_n^{\varepsilon_k})}] \geq \delta_0.$$

For each $k \in \mathbb{N}$, let $x_k \in H_{\varepsilon_k}$ be such that

$$\mathbb{E}_{x_k}[1 - e^{-L(\beta_n^{\varepsilon_k})}] \geq \frac{\delta_0}{2}. \tag{4.38}$$

Since, the law of $(Z_{(k)}, Y_{(k)}, L_{(k)})$ under $\mathbb{P}$ is the same as the law of (Z, Y, L) under $\mathbb{P}_{x_k}$, (4.38) is equivalent to the inequality

$$\mathbb{E}[1 - e^{-L_{(k)}(\beta^{\varepsilon_k}_{(k),n})}] \geq \frac{\delta_0}{2}. \tag{4.39}$$

The scaling relations in (4.12) and (4.24) show that

$$\mathbb{E}[1 - e^{-L_{(k)}(\beta^{\varepsilon_k}_{(k),n})}] = \mathbb{E}[1 - e^{-\varepsilon_k L^k(\beta_n^{k,1})}]. \tag{4.40}$$

Moreover, since $Z^k(0) = x_k/\varepsilon_k \in H_1$, it follows that $\langle Z^k(t), \vec{\mathbf{v}}\rangle \in [2^{-n}, 2^n]$ for $t \in [0, \zeta_{k,n}]$, Therefore, there exist $0 < r < 2^{-n}$ and $R > 2^n$ such that $\zeta_{k,n} \le \theta^k_{r,R}$, where $\theta^k_{r,R}$ is defined in (4.13). As a result, we conclude that

$$\mathbb{E}[1 - e^{-\varepsilon_k L^k(\zeta^{k,n})}] \le \mathbb{E}[1 - e^{-\varepsilon_k L^k(\theta^k_{r,R})}] \to 0 \qquad \text{as } k \to \infty,$$

where the last limit holds due to the weak convergence $\varepsilon_k L^k(\theta^k_{r,R}) \Rightarrow 0$ established in Corollary 4.8, and the fact that $x \mapsto 1 - e^{-x}$ is a bounded continuous function. When combined with (4.40), this contradicts (4.38) and thus proves the lemma. $\square$

We are now in a position to complete the proof of Proposition 4.2.

Proof of Proposition 4.2. First, observe that by Lemma 4.9, there exists $C > 0$ and $K < \infty$ such that for all $k \ge K$, the relation

$$\inf_{x \in H_{\varepsilon_k}} \mathbb{E}_x[(1 - e^{-L(\beta_1^{\varepsilon_k})})\mathbb{I}_{\{\tau^0 < \tau^1\}}] \ge \frac{C}{2}\varepsilon_k$$

is satisfied. Together with the fact that $Z(\tau^{\varepsilon_k}) \in H_{\varepsilon_k}$ and, for any $x \in H_{\varepsilon_k}$, $\mathbb{P}_x$-a.s.,

$$\langle Z(\beta^{\varepsilon_k}_{n-1}), \vec{\mathbf{v}}\rangle \le 2^{n-1}\varepsilon_k, \tag{4.41}$$

implies that for all k large enough so that $\varepsilon_k < 2^{-(n-1)}\varepsilon_0$ and for $m = 1, \ldots, n$,

$$\begin{aligned} &\mathbb{E}_{Z(\tau^{\varepsilon_k})}[e^{-L(\beta^{\varepsilon_k}_{m-1})}\mathbb{E}_{Z(\beta^{\varepsilon_k}_{m-1})}[(1 - e^{-L(\beta_1^{\varepsilon_k})})\mathbb{I}_{\{\tau^0<\tau^1\}}]] \\ &\qquad \ge \frac{C}{2}\mathbb{E}_{Z(\tau^{\varepsilon_k})}[e^{-L(\beta^{\varepsilon_k}_{m-1})}\langle Z(\beta^{\varepsilon_k}_{m-1}), \vec{\mathbf{v}}\rangle]. \end{aligned} \tag{4.42}$$

When combined with (4.26) and (4.27), this shows that

$$\begin{aligned} &\mathbb{E}_0[\mathbb{E}_{Z(\tau^{\varepsilon_k})}[(1 - e^{-L(\tau^0)})\mathbb{I}_{\{\tau^0<\tau^1\}}]] \\ &\qquad \ge \frac{C}{2}\sum_{m=1}^n \mathbb{E}_{Z(\tau^{\varepsilon_k})}[e^{-L(\beta^{\varepsilon_k}_{m-1})}\langle Z(\beta^{\varepsilon_k}_{m-1}), \vec{\mathbf{v}}\rangle]. \end{aligned} \tag{4.43}$$

Each summand on the right-hand side can be rewritten in the more convenient form

$$\begin{aligned} &\mathbb{E}_{Z(\tau^{\varepsilon_k})}[e^{-L(\beta^{\varepsilon_k}_{m-1})}\langle Z(\beta^{\varepsilon_k}_{m-1}), \vec{\mathbf{v}}\rangle] \\ &\quad = \mathbb{E}_{Z(\tau^{\varepsilon_k})}[\langle Z(\beta^{\varepsilon_k}_{m-1}), \vec{\mathbf{v}}\rangle] - \mathbb{E}_{Z(\tau^{\varepsilon_k})}[(1 - e^{-L(\beta^{\varepsilon_k}_{m-1})})\langle Z(\beta^{\varepsilon_k}_{m-1}), \vec{\mathbf{v}}\rangle]. \end{aligned}$$

Since $b \equiv 0$, Lemma 4.4 and the uniform bound (4.41) show that $\langle Z, \vec{\mathbf{v}} \rangle$ is a martingale on $[0, \beta_n^{\varepsilon_k}]$. In addition, because $\beta_{m-1}^{\varepsilon_k} \le \beta_n^{\varepsilon_k}$ and $\langle Z(\tau^{\varepsilon_k}), \vec{\mathbf{v}} \rangle = \varepsilon_k$, it follows that

$$\mathbb{E}_0[\mathbb{E}_{Z(\tau^{\varepsilon_k})}[\langle Z(\beta_{m-1}^{\varepsilon_k}), \vec{\mathbf{v}} \rangle]] = \mathbb{E}_0[\varepsilon_k] = \varepsilon_k.$$

Furthermore, by (4.41), Lemma 4.10 and the bounded convergence theorem, we have for any $n \in \mathbb{N}$ and $m = 1, \dots, n$,

$$\begin{aligned}\limsup_{k\to\infty} \frac{1}{\varepsilon_k} \mathbb{E}_0[\mathbb{E}_{Z(\tau^{\varepsilon_k})}[(1 - e^{-L(\beta_{m-1}^{\varepsilon_k})}) \langle Z(\beta_{m-1}^{\varepsilon_k}), \vec{\mathbf{v}} \rangle]] \\ \le 2^{n-1} \lim_{k\to\infty} \mathbb{E}_0\Big[\sup_{x \in H_{\varepsilon_k}} \mathbb{E}_x[1 - e^{-L(\beta_{m-1}^{\varepsilon_k})}]\Big] = 0.\end{aligned}$$

Combining the last three assertions, we see that for every $n \in \mathbb{N}$ and $m = 1, \dots, n$,

$$\liminf_{k\to\infty} \frac{1}{\varepsilon_k} \mathbb{E}_0[\mathbb{E}_{Z(\tau^{\varepsilon_k})}[e^{-L(\beta_{m-1}^{\varepsilon_k})} \langle Z(\beta_{m-1}^{\varepsilon_k}), \vec{\mathbf{v}} \rangle]] = 1.$$

Together with (4.43), this shows that for every $n \in \mathbb{N}$,

$$\liminf_{k\to\infty} \frac{1}{\varepsilon_k} \mathbb{E}_0[\mathbb{E}_{Z(\tau^{\varepsilon_k})}[(1 - e^{-L(\tau^0)}) \mathbb{I}_{\{\tau^0 < \tau^1\}}]] \ge \frac{nC}{2}.$$

Taking the limit as $n \to \infty$, we obtain (4.21), thus completing the proof of the proposition. $\square$

4.2. *The general drift case.* In this section, we establish Theorem 4.1. Specifically, we use a Girsanov transformation to generalize the case of zero drift, established in Proposition 4.2, to arbitrary Lipschitz drifts with linear growth, as specified in property 3 of Definition 2.5. As before, let Z be the unique strong solution to the Class $\mathcal{A}$ SDER, which exists by Theorem 2.7, and let τ^1 be the first hitting time to H_1, as defined in (4.2). We begin with a simple lemma that shows that τ^1 is finite with positive $\mathbb{P}_0$ probability.

LEMMA 4.11. *We have*

$$\mathbb{P}_0(\tau^1 < \infty) > 0. \tag{4.44}$$

Moreover, if $\inf_{x:\langle x, \vec{\mathbf{v}} \rangle \le 1} \langle b(x), \vec{\mathbf{v}} \rangle \ge 0$, *then*

$$\mathbb{P}_0(\tau^1 < \infty) = 1. \tag{4.45}$$

PROOF. Recall the definition of X and M given in (2.4) and (2.12). By Theorem 2.7, we know that $\mathbb{P}$-a.s., Z satisfies the ESP for X. Hence, by Lemma 2.8 it follows that $\widehat{Z} = \Gamma_1(\widehat{X})$, where Γ_1 is the 1-dimensional Skorokhod map and, for $H = Z, M, X$, we define $\widehat{H} \doteq \langle H, \vec{\mathbf{v}} \rangle$. Let $T(t) \doteq \inf\{s \ge$

$0 : \langle \widehat{M} \rangle_s > t\}$. Then, due to the uniform ellipticity of a, T is strictly increasing and, since $\widehat{M}$ is a continuous martingale, $\widehat{M}(T(\cdot))$ is a 1-dimensional Brownian motion. In turn, this implies $\widehat{Z}$ is a one-dimensional reflected Brownian motion with drift

$$\int_0^t \langle b(Z(T(s))), \vec{\mathbf{v}}\rangle \, dT(s) = \int_0^t \langle b(Z(T(s))), \vec{\mathbf{v}}\rangle \frac{1}{\vec{\mathbf{v}}^T a(Z(s))\vec{\mathbf{v}}} \, ds.$$

Since $\langle b(x), \vec{\mathbf{v}}\rangle / \vec{\mathbf{v}}^T a(x) \vec{\mathbf{v}}$ is continuous on G, there exists $\kappa \in (-\infty, \infty)$ such that

$$\frac{\langle b(x), \vec{\mathbf{v}}\rangle}{\vec{\mathbf{v}}^T a(x)\vec{\mathbf{v}}} > \kappa \qquad \text{for all } x \in G, \langle x, \vec{\mathbf{v}}\rangle \le 1.$$

Consider the process $\tilde{X}$ defined by $\tilde{X}(t) \doteq \kappa t + M(T(t))$ for $t \in [0, \infty)$, and let $\tilde{Z} \doteq \Gamma_1(\tilde{X})$ be a one-dimensional reflected Brownian motion with constant drift κ. Then $\widehat{X}(T(t)) - \widehat{X}(T(s)) \ge \tilde{X}(t) - \tilde{X}(s)$ for every $0 \le s \le t$, and so the comparison principle for Γ_1 (see, e.g., equation (4.1) in Lemma 4.1 of [20]) shows that $\widehat{Z}(T(t)) \ge \tilde{Z}(t)$ for every $t \in [0, \widehat{\tau}^1]$, where

$$\widehat{\tau}^1 \doteq \inf\{t > 0 : \widehat{Z}(T(t)) = 1\}.$$

Since $T(\widehat{\tau}^1) = \tau^1$, it follows that

$$\mathbb{P}_0(\widehat{Z}(T(t) \wedge \tau^1) \ge \tilde{Z}(t \wedge \widehat{\tau}^1) \text{ for all } t \ge 0) = 1.$$

Since T is strictly increasing, we have $\tau^1 = \infty$ if and only if $\widehat{\tau}^1 = \infty$. Therefore, on the set $\{\tau^1 = \infty\}$, we must have

$$\tilde{Z}(t) \le \widehat{Z}(T(t)) < 1 \qquad \text{for all } t \in [0, \infty).$$

However, $\tilde{Z}$ will hit 1 with positive $\mathbb{P}_0$ probability, and in fact will hit 1 $\mathbb{P}_0$-a.s. if $\kappa \ge 0$ (see, e.g., page 197 of [19]), and so the same is true of $\hat{Z}(T(\cdot))$. This implies both (4.44) and (4.45), and so the proof of the lemma is complete. $\square$

Proof of Theorem 4.1. The uniform ellipticity of $a(\cdot)$ ensures that $a^{-1}(\cdot)$ exists. Let $\mu \doteq -\sigma^T a^{-1} b$, note that $\mu^T \mu = b^T a b$, and define

$$D(t) \doteq \exp\left\{\int_0^t \mu(Z(s))\, dB(s) - \frac{1}{2}\int_0^t b^T(Z(s)) a(Z(s)) b(Z(s))\, ds\right\} \tag{4.46}$$

for $t \in [0, \infty)$. Property 3 of Definition 2.5 guarantees that μ has at most linear growth, and so, as is well-known, $\{D(t), \mathcal{F}_t\}$ is a martingale (see, e.g., Corollary 5.16 of [19]).

Fix $T < \infty$. Define a new probability measure $\mathbb{Q}_0$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_T\})$ by setting

$$\mathbb{Q}_0(A) = \mathbb{E}[D(T)\mathbb{I}_A] \qquad \text{for } A \in \mathcal{F}_T.$$

Define

$$\tilde{B}(t) \doteq B(t) + \int_0^t \sigma^T(Z(s))a^{-1}(Z(s))b(s)\,ds, \qquad t \in [0,T].$$

By Girsanov's theorem (see Theorem 5.1 of [19]), under $\mathbb{Q}_0$, $\{\tilde{B}_t, \mathcal{F}_t\}_{t\in[0,T]}$ is a Brownian motion and

$$Z(t) = \int_0^t \sigma(Z(s))\,d\tilde{B}(s) + Y(t), \qquad t \in [0,T],$$

where (Z,Y) satisfy the ESP pathwise for $Z-Y$. Since, under $\mathbb{Q}_0$, Z is the solution to a Class $\mathcal{A}$ SDER with no drift, by Proposition 4.2, it follows that

$$\mathbb{Q}_0(L(\tau^1) < \infty, \tau^1 \le T) = 0.$$

Since $\mathbb{P}_0 \ll \mathbb{Q}_0$ [with $d\mathbb{P}_0/d\mathbb{Q}_0 = D^{-1}(T)$ on $\mathcal{F}_T$], this implies

$$\mathbb{P}_0(L(\tau^1) < \infty, \tau^1 \le T) = 0.$$

Since $T < \infty$ is arbitrary, sending $T \to \infty$ (along a countable sequence), we conclude that

$$\mathbb{P}_0(L(\tau^1) < \infty, \tau^1 < \infty) = 0.$$

However, $\mathbb{P}_0(\tau^1 < \infty) > 0$ by Lemma 4.11. Hence, $\mathbb{P}_0(L(\tau^1) = \infty, \tau^1 < \infty) > 0$, which in turn implies that there exists $T < \infty$ such that $\mathbb{P}_0(L(T) = \infty) > 0$, which proves Theorem 4.1. In addition, note that if $\inf_{x\in G:\langle x,\vec{\mathbf{v}}\rangle\le 1}\langle b(x), \vec{\mathbf{v}}\rangle \ge 0$, then $\mathbb{P}_0(\tau^1 < \infty) = 1$ and so we in fact have $\mathbb{P}_0(L(\tau^1) = \infty) = 1$. $\square$

4.3. *The semimartingale property for Z.* Recall from Theorem 2.7 that the process Z has the decomposition $Z = M + A$, where

$$M = \int_0^\cdot \sigma(Z(s))\,dB(s), \qquad A = \int_0^\cdot b(Z(s))\,ds + Y, \tag{4.47}$$

and Y is the constraining term associated with the ESP. M is clearly a (local) martingale, but Theorem 4.1 shows that Y is not $\mathbb{P}$-a.s. of finite variation on bounded intervals. However, as mentioned earlier, Theorem 4.1 does not immediately imply that Z is not a semimartingale because we do not know a priori that the above decomposition must be the Doob decomposition of Z if it were a semimartingale. In Proposition 4.12 below, we show that the latter statement is indeed true, thus showing that Z is not a semimartingale.

PROPOSITION 4.12. *If Z were a semimartingale, then its Doob decomposition must be $Z = M + A$.*

PROOF. Suppose that Z is a semimartingale, and let its (unique) Doob decomposition take the form

$$Z = \tilde{M} + \tilde{A},$$

where $\tilde{M}$ is an $\{\mathcal{F}_t\}$-adapted continuous local martingale and $\tilde{A}$ is an $\{\mathcal{F}_t\}$-adapted continuous, process with $\mathbb{P}$-a.s. finite variation on bounded intervals.

Fix $R < \infty$ and let $\theta_R \doteq \inf\{t \geq 0 : |M(t)| \geq R\}$. For each $\varepsilon > 0$, define two sequences of stopping times $\{\tau_n^\varepsilon\}_{n\in\mathbb{N}}$ and $\{\xi_n^\varepsilon\}_{n\in\mathbb{N}}$ as follows: $\xi_0^\varepsilon \doteq 0$ and for $n \in \mathbb{N}$, let

$$\tau_n^\varepsilon \doteq \inf\{t \geq \xi_{n-1}^\varepsilon : Z(t) \in H_\varepsilon\} \wedge \theta_R,$$
$$\xi_n^\varepsilon \doteq \inf\{t \geq \tau_n^\varepsilon : Z(t) \in H_{\varepsilon/2}\} \wedge \theta_R.$$

(For notational conciseness, we have suppressed the dependence of these stopping times on R.) By uniqueness of the Doob decomposition, clearly $Z(\cdot \wedge \xi_n^\varepsilon) - Z(\cdot \wedge \tau_n^\varepsilon)$ is an $\{\mathcal{F}_t\}$-adapted semimartingale, with Doob decomposition

$$Z(t \wedge \xi_n^\varepsilon) - Z(t \wedge \tau_n^\varepsilon) = \tilde{M}(t \wedge \xi_n^\varepsilon) - \tilde{M}(t \wedge \tau_n^\varepsilon) + \tilde{A}(t \wedge \xi_n^\varepsilon) - \tilde{A}(t \wedge \tau_n^\varepsilon)$$

On the other hand, due to the identity $Z = M + A = \tilde{M} + \tilde{A}$, we also have

$$Z(t \wedge \xi_n^\varepsilon) - Z(t \wedge \tau_n^\varepsilon) = M(t \wedge \xi_n^\varepsilon) - M(t \wedge \tau_n^\varepsilon) + A(t \wedge \xi_n^\varepsilon) - A(t \wedge \tau_n^\varepsilon).$$

Since M is an $\{\mathcal{F}_t\}$-adapted continuous (local) martingale, and M is uniformly bounded on $[0, \theta_R]$, the stopped processes $M(\cdot \wedge \xi_n^\varepsilon)$ and $M(\cdot \wedge \tau_n^\varepsilon)$ are $\{\mathcal{F}_t\}$-adapted continuous martingales. Hence, $M(\cdot \wedge \xi_n^\varepsilon) - M(\cdot \wedge \tau_n^\varepsilon)$ is also an $\{\mathcal{F}_t\}$-adapted continuous martingale. Moreover, Theorem 2.7 implies that $Y(\cdot \wedge \xi_n^\varepsilon) - Y(\cdot \wedge \tau_n^\varepsilon)$ has $\mathbb{P}$-a.s. finite variation on each bounded time interval. Since $A = Y + \int_0^\cdot b(Z(s))\, ds$, $A(\cdot \wedge \xi_n^\varepsilon) - A(\cdot \wedge \tau_n^\varepsilon)$ also has $\mathbb{P}$-a.s. finite variation on each bounded time interval. By uniqueness of the Doob decomposition, we conclude that for every $\varepsilon > 0$ and $t \in [0, \infty)$,

$$M(t \wedge \xi_n^\varepsilon) - M(t \wedge \tau_n^\varepsilon) = \tilde{M}(t \wedge \xi_n^\varepsilon) - \tilde{M}(t \wedge \tau_n^\varepsilon).$$

Summing over $n \in \mathbb{N}$ on both sides of the last equation, we obtain

$$\sum_{n=1}^{\infty} (M(t \wedge \xi_n^\varepsilon) - M(t \wedge \tau_n^\varepsilon)) = \sum_{n=1}^{\infty} (\tilde{M}(t \wedge \xi_n^\varepsilon) - \tilde{M}(t \wedge \tau_n^\varepsilon)). \tag{4.48}$$

On the other hand, because $\mathbb{P}$-a.s., $M(0) = 0$ and $\xi_n^\varepsilon \to \theta_R$ as $n \to \infty$, we can write $M(t \wedge \theta_R)$ as a telescopic sum:

$$M(t \wedge \theta_R) = \sum_{n=1}^{\infty} (M(t \wedge \xi_n^\varepsilon) - M(t \wedge \xi_{n-1}^\varepsilon)), \qquad t \in [0, \infty).$$

Next, observe that

$$\begin{aligned} M(t\wedge\theta_R) &- \sum_{n=1}^{\infty}(M(t\wedge\xi_n^{\varepsilon}) - M(t\wedge\tau_n^{\varepsilon})) \\ &= \sum_{n=1}^{\infty}(M(t\wedge\tau_n^{\varepsilon}) - M(t\wedge\xi_{n-1}^{\varepsilon})) \\ &= \int_0^t \sum_{n=1}^{\infty}\mathbb{I}_{(\xi_{n-1}^{\varepsilon},\tau_n^{\varepsilon}]}(s)\,dM(s) \\ &= \int_0^t \sum_{n=1}^{\infty}\mathbb{I}_{(\xi_{n-1}^{\varepsilon},\tau_n^{\varepsilon}]}(s)\mathbb{I}_{[0,\varepsilon]}(\langle\vec{\mathbf{v}},Z(s)\rangle)\,dM(s), \end{aligned}$$

where the last equality holds because $\langle\vec{\mathbf{v}},Z(s)\rangle\le\varepsilon$ for $s\in(\xi_{n-1}^{\varepsilon},\tau_n^{\varepsilon}]$. When combined with Doob's maximal martingale inequality, this yields

$$\begin{aligned} \mathbb{E}&\left[\sup_{s\in[0,t]}\left|M(s\wedge\theta_R)-\sum_{n=1}^{\infty}(M(s\wedge\xi_n^{\varepsilon})-M(s\wedge\tau_n^{\varepsilon}))\right|^2\right] \\ &\le 4\mathbb{E}\left[\left|M(t\wedge\theta_R)-\sum_{n=1}^{\infty}(M(t\wedge\xi_n^{\varepsilon})-M(t\wedge\tau_n^{\varepsilon}))\right|^2\right] \\ &= 4\mathbb{E}\left[\left|\int_0^t\sum_{n=1}^{\infty}\mathbb{I}_{(\xi_{n-1}^{\varepsilon},\tau_n^{\varepsilon}]}(s)\mathbb{I}_{[0,\varepsilon]}(\langle\vec{\mathbf{v}},Z(s)\rangle)\,dM(s)\right|^2\right] \\ &\le 4\mathbb{E}\left[\int_0^t\mathbb{I}_{[0,\varepsilon]}(\langle\vec{\mathbf{v}},Z(s)\rangle)|a(Z(s))|\,ds\right]. \end{aligned}$$

By Assumption 3, a is bounded on the set $\{x:\langle\vec{\mathbf{v}},x\rangle\le\varepsilon\}$. Hence, an application of the bounded convergence theorem shows that

$$\lim_{\varepsilon\to0}\mathbb{E}\left[\int_0^t\mathbb{I}_{\{\langle\vec{\mathbf{v}},Z(s)\rangle\le\varepsilon\}}|a(Z(s))|\,ds\right]=|a(0)|\mathbb{E}\left[\int_0^t\mathbb{I}_{\{\langle\vec{\mathbf{v}},Z(s)\rangle=0\}}\,ds\right]=0,$$

where the last equality is a consequence of the fact that $\langle\vec{\mathbf{v}},Z\rangle$ is a uniformly elliptic one-dimensional reflected diffusion (see Lemma 2.8) and consequently spends zero Lebesgue time at the origin (see, e.g., page 90 of [15]).

An exactly analogous argument, with $\tilde{\theta}_R\doteq\inf\{t\ge0:|\tilde{M}|(t)\ge R\}$ and $\tilde{\xi}_n^{\varepsilon},\tilde{\tau}_n^{\varepsilon}$ defined in a fashion analogous to $\xi_n^{\varepsilon},\tau_n^{\varepsilon}$, but with θ_R replaced by $\tilde{\theta}_R$, shows that

$$\lim_{\varepsilon\to0}\mathbb{E}\left[\sup_{s\in[0,t]}\left|\tilde{M}(s\wedge\tilde{\theta}_R)-\sum_{n=1}^{\infty}(\tilde{M}(s\wedge\tilde{\xi}_n^{\varepsilon})-\tilde{M}(s\wedge\tilde{\tau}_n^{\varepsilon}))\right|^2\right]$$

$$
\begin{aligned}
&\leq \lim_{\varepsilon\to 0} 4J^2 \sum_{i=1}^J \mathbb{E}\left[\int_0^t \mathbb{I}_{[0,\varepsilon]}(\langle \vec{\mathbf{v}}, Z(s)\rangle)\, d\langle \tilde{M}_i\rangle(s)\right] \\
&= 4J^2 \sum_{i=1}^J \mathbb{E}\left[\int_0^t \mathbb{I}_{\{0\}}(\langle \vec{\mathbf{v}}, Z(s)\rangle)\, d\langle \tilde{M}_i\rangle(s)\right] \\
&= 4J^2 \sum_{i=1}^J \mathbb{E}\left[\int_0^t \mathbb{I}_{\{0\}}(Z_i(s))\, d\langle \tilde{M}_i\rangle(s)\right],
\end{aligned}
$$

where the last equality uses the property that $Z_i(s)=0$ for every $i=1,\dots,J$ if and only if $\langle \vec{\mathbf{v}}, Z(s)\rangle = 0$ (see property 2 of Definition 2.5). Due to the assumption that $\tilde{Z}_i$ is a semimartingale with decomposition $\tilde{M}_i + \tilde{A}_i$, the occupation times formula for continuous semimartingales (see, e.g., Corollary 1.6 in Chapter VI of [25]) and the fact that the set $\{x : x_i = 0\}$ has zero Lebesgue measure, we have, $\mathbb{P}$-a.s., for $i=1,\dots,J$,

$$
\int_0^t \mathbb{I}_{\{0\}}(Z_i(s))\, d\langle \tilde{M}_i\rangle(s) = \int_0^t \mathbb{I}_{\{0\}}(Z_i(s))\, d\langle Z_i\rangle(s) = 0.
$$

Combining the last four displays with (4.48), we conclude that $M(t\wedge\theta_R) = \tilde{M}(t\wedge\tilde{\theta}_R)$, $\mathbb{P}_0$-a.s., for every $t\geq 0$. This in turn implies that $\theta_R = \tilde{\theta}_R$ $\mathbb{P}_0$-a.s. Sending $R\to\infty$ and invoking the continuity of both M and $\tilde{M}$, we conclude that $M=\tilde{M}$ $\mathbb{P}_0$-a.s. In turn, this implies $A=\tilde{A}$, thus completing the proof of the theorem. □

The proof of Theorem 3.1 is now a simple consequence of Theorem 4.1 and Proposition 4.12.

Proof of Theorem 3.1. If Z were a semimartingale under $\mathbb{P}_0$, then by Proposition 4.12, $Z=M+A$ is the Doob decomposition for Z. In particular, this implies that $\mathbb{P}_0(L(T)<\infty)=1$ for every $T\in[0,\infty)$, where recall that $L(T)=\mathrm{Var}_{[0,T]}Y$. However, this contradicts the assertion of Theorem 4.1 that there exists $T<\infty$ such that $\mathbb{P}_0(L(T)=\infty)>0$. Thus, we conclude that Z is not a semimartingale. □

Remark 4.13. It is natural to expect that similar, but somewhat more involved, arguments could be used to show that the semimartingale property fails to hold for a more general class of reflected diffusions in the nonnegative orthant, in particular those that arise as approximations of generalized processor sharing networks (rather than just a single station, as considered in [23, 24]). Such diffusions would satisfy properties 1, 2 and 4 of Definition 2.5 but would have more complicated $\mathcal{V}$-sets (see [10] for a description of the ESP associated with such a network). This is a subject for future work.

5. Dirichlet process characterization. This section is devoted to the proof of Theorem 3.5. Specifically, here we only assume that $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ satisfy Assumptions 1 and 3, and let $(Z_t, B_t), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$ be a Markov process that is a weak solution to the associated SDER that satisfies Assumption 2 for some constants $p > 1, q \geq 2$ and $K_T < \infty, T \in (0, \infty)$. As usual, let X be as defined in (2.4), and let $Y = Z - X$, so that we can write

$$Z(t) = Z(0) + \int_0^t b(Z(s))\,ds + \int_0^t \sigma(Z(s))\,dB(s) + Y(t), \qquad t \in [0, \infty).$$

Note that $\int_0^\cdot b(Z(s))\,ds$ is a process of bounded variation, and therefore of bounded p-variation for any $p > 1$ by Remark 3.4. As a result, in order to establish Theorem 3.5, it suffices to show that under $\mathbb{P}$-a.s., Y has zero p-variation.

In Section 5.1, we first show that it suffices to establish a localized version (5.3) of the zero p-variation condition on Y. This is then used to prove Theorem 3.5 in Section 5.2.

5.1. *Localization.* Fix $T > 0$, let $\{\pi^n, n \geq 1\}$ be a sequence of partitions of $[0, T]$ such that $\Delta(\pi^n) \to 0$ as $n \to \infty$. As mentioned above, to prove Theorem 3.5 we need to establish the following result:

$$\sum_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^p \stackrel{(\mathbb{P})}{\to} 0 \qquad \text{as } \Delta(\pi^n) \to \infty. \tag{5.1}$$

For each $m \in (0, \infty)$, let

$$\zeta^m \doteq \inf\{t > 0 : |Z(t)| \geq m\}. \tag{5.2}$$

It is easy to see that $\mathbb{P}$-a.s., $\zeta^m \to \infty$ as $m \to \infty$. We now show that the localized version, (5.3) below, is equivalent to (5.1).

LEMMA 5.1. *The result (5.1) holds if and only if for each $m \in (0, \infty)$,*

$$\sum_{t_i \in \pi^n} |Y(t_i \wedge \zeta^m) - Y(t_{i-1} \wedge \zeta^m)|^p \stackrel{(\mathbb{P})}{\to} 0 \qquad \textit{as } \Delta(\pi^n) \to 0. \tag{5.3}$$

PROOF. First, assume (5.3) holds for every $m \in (0, \infty)$. Then, for every $m \in (0, \infty)$ and $\delta > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sum_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^p \geq \delta\right) \\
&\qquad \leq \mathbb{P}\left(\sum_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^p \geq \delta, \zeta^m > T\right) + \mathbb{P}(\zeta^m \leq T)
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\biggl(\sum_{t_i\in\pi^n} |Y(t_i\wedge\zeta^m) - Y(t_{i-1}\wedge\zeta^m)|^p \geq \delta, \zeta^m > T\biggr) + \mathbb{P}(\zeta^m \leq T) \\
&\leq \mathbb{P}\biggl(\sum_{t_i\in\pi^n} |Y(t_i\wedge\zeta^m) - Y(t_{i-1}\wedge\zeta^m)|^p \geq \delta\biggr) + \mathbb{P}(\zeta^m \leq T).
\end{aligned}
$$

Taking limits as $\Delta(\pi^n)\to 0$, the first term on the right-hand side vanishes due to (5.3). Next, sending $m\to\infty$, and using the fact that $\zeta^m\to\infty$ $\mathbb{P}$-a.s., the second term also vanishes, and so we obtain (5.1). This proves the "if" part of the result.

In order to prove the converse result, suppose (5.1) holds. Let $\theta_n^m \doteq \sup\{t_i\in\pi^n : t_i\leq\zeta^m\}$, where $\theta_n^m\doteq T$ if the latter set is empty. Then

$$
\begin{aligned}
&\sum_{t_i\in\pi^n} |Y(t_i\wedge\zeta^m) - Y(t_{i-1}\wedge\zeta^m)|^p \\
&\qquad \leq \sum_{t_i\in\pi^n} |Y(t_i) - Y(t_{i-1})|^p + |Y(\zeta^m\wedge T) - Y(\theta_n^m)|^p.
\end{aligned}
$$

Taking limits as $\Delta(\pi^n)\to 0$, $\mathbb{P}$-a.s the last term vanishes since $|\zeta^m\wedge T - \theta_n^m| \leq \Delta(\pi^n)$ and Y is continuous. Therefore, (5.3) follows from (5.1). $\square$

5.2. *The decomposition result.* For each $\varepsilon>0$, recursively define two sequences of stopping times $\{\tau_n^\varepsilon\}_{n\in\mathbb{N}}$ and $\{\xi_n^\varepsilon\}_{n\in\mathbb{N}}$ as follows: $\xi_0^\varepsilon\doteq 0$ and for $n\in\mathbb{N}$,

$$
\begin{aligned}
\tau_n^\varepsilon &\doteq \inf\{t\geq \xi_{n-1}^\varepsilon : d(Z(t),\mathcal{V}) = \varepsilon\}, \\
\xi_n^\varepsilon &\doteq \inf\{t\geq \tau_n^\varepsilon : d(Z(t),\mathcal{V}) = \varepsilon/2\}.
\end{aligned} \tag{5.4}
$$

For each $\varepsilon>0$, we have the decomposition

$$
\begin{aligned}
\sum_{t_i\in\pi^n} |Y(t_i) - Y(t_{i-1})|^p &= \sum_{t_i\in\pi^n}\sum_{k=1}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{(\tau_k^\varepsilon,\xi_k^\varepsilon)}(t_{i-1}) \\
&\quad + \sum_{t_i\in\pi^n}\sum_{k=0}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon]}(t_{i-1}).
\end{aligned}
$$

Therefore, for any given $\delta>0$, we have

$$
\begin{aligned}
&\mathbb{P}\biggl(\sum_{t_i\in\pi^n} |Y(t_i) - Y(t_{i-1})|^p > \delta\biggr) \\
&\qquad \leq \mathbb{P}\biggl(\sum_{t_i\in\pi^n}\sum_{k=1}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\tau_k^\varepsilon,\xi_k^\varepsilon)}(t_{i-1}) > \frac{\delta}{2}\biggr) \\
&\qquad\quad + \mathbb{P}\biggl(\sum_{t_i\in\pi^n}\sum_{k=0}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon)}(t_{i-1}) > \frac{\delta}{2}\biggr).
\end{aligned} \tag{5.5}
$$

Under additional uniform boundedness assumptions on b and σ, the proof of (5.1) is essentially a consequence of the following two lemmas, which provide estimates on the two terms on the right-hand side of (5.5).

LEMMA 5.2. *Suppose b and σ are uniformly bounded. Then, for each $\varepsilon > 0$,*

$$\lim_{\Delta(\Pi_n)\to 0} \mathbb{P}\left(\sum_{t_i\in\pi^n}\sum_{k=1}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\tau_k^\varepsilon,\xi_k^\varepsilon)}(t_{i-1}) > \frac{\delta}{2}\right) = 0. \tag{5.6}$$

PROOF. Fix $\varepsilon > 0$, $n \in \mathbb{N}$, and let

$$\Omega_n^\varepsilon \doteq \left\{Z(t) \notin \mathcal{V},\ \forall t \in \bigcup_{k\in\mathbb{N}:\xi_k^\varepsilon \le T} [\xi_k^\varepsilon, \xi_k^\varepsilon + \Delta(\pi^n)]\right\}.$$

Also, define

$$N^\varepsilon \doteq \inf\{k \ge 0 : \text{either } \tau_k^\varepsilon > T \text{ or } \xi_k^\varepsilon > T\}.$$

Observe that $\mathbb{P}$-a.s., $N^\varepsilon < \infty$ since Z has continuous sample paths and therefore crosses the levels $\{z \in G : d(z,\mathcal{V}) = \varepsilon\}$ and $\{z \in G : d(z,\mathcal{V}) = \varepsilon/2\}$ at most a finite number of times in the interval $[0,T]$. The continuity of Z also implies that for each $\varepsilon > 0$,

$$\mathbb{P}(\Omega_n^\varepsilon) \to 1 \qquad \text{as } \Delta(\pi^n) \to 0. \tag{5.7}$$

On the set Ω_n^ε, we have

$$\begin{aligned}
&\sum_{t_i\in\pi^n}\sum_{k=1}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\tau_k^\varepsilon,\xi_k^\varepsilon)}(t_{i-1}) \\
&\qquad \le \max_{t_i\in\pi^n} |Y(t_i) - Y(t_{i-1})|^{p-1} \sum_{t_i\in\pi^n}\sum_{k=1}^{\infty} L(t_{i-1},t_i]\mathbb{I}_{[\tau_k^\varepsilon,\xi_k^\varepsilon)}(t_{i-1}) \\
&\qquad = \max_{t_i\in\pi^n} |Y(t_i) - Y(t_{i-1})|^{p-1} \sum_{t_i\in\pi^n}\sum_{k=1}^{\infty} L(t_{i-1},t_i]\mathbb{I}_{[\tau_k^\varepsilon,\xi_k^\varepsilon)}(t_{i-1}) \\
&\qquad \le \max_{t_i\in\pi^n} |Y(t_i) - Y(t_{i-1})|^{p-1} \sum_{k=1}^{\infty} L(\tau_k^\varepsilon \wedge T, (\xi_k^\varepsilon + \Delta(\pi^n)) \wedge T].
\end{aligned} \tag{5.8}$$

By definition, $\mathbb{P}$-a.s. (Z,Y) satisfy the ESP for X. Therefore, by Lemma A.1, $\mathbb{P}$-a.s., for each $k \in \mathbb{N}$, $(Z(\tau_k^\varepsilon \wedge T + \cdot), Y(\tau_k^\varepsilon \wedge T + \cdot) - Y(\tau_k^\varepsilon \wedge T))$ solve the ESP for $Z(\tau_k^\varepsilon \wedge T) + X(\tau_k^\varepsilon \wedge T + \cdot) - X(\tau_k^\varepsilon \wedge T)$. On Ω_n^ε, Z is away from $\mathcal{V}$ on $[\tau_k^\varepsilon \wedge T, (\xi_k^\varepsilon + \Delta(\pi^n)) \wedge T]$ for each $k \ge 1$, and hence by Theorem 2.9 of

[22] it follows that $L(\tau_k^\varepsilon \wedge T, (\xi_k^\varepsilon + \Delta(\pi^n)) \wedge T] < \infty$. Together with the fact that $\mathbb{P}$-a.s. $N^\varepsilon < \infty$, this implies that

$$\sum_{k=1}^\infty L(\tau_k^\varepsilon \wedge T, (\xi_k^\varepsilon + \Delta(\pi^n)) \wedge T] < \infty \qquad \mathbb{P}\text{-a.s. on } \Omega_n^\varepsilon.$$

On the other hand, since Y is continuous on $[0,T]$ and $p > 1$, we have

$$\max_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^{p-1} \to 0 \qquad \text{as } \Delta(\pi^n) \to 0.$$

Combining the above two displays with (5.7), we conclude that for every $\delta > 0$, as $\Delta(\pi^n) \to 0$,

$$\mathbb{P}\left(\max_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^{p-1} \sum_{k=1}^\infty L(\tau_k^\varepsilon \wedge T, (\xi_k^\varepsilon + \Delta(\pi^n)) \wedge T] > \frac{\delta}{2}\right) \to 0.$$

Together with (5.8), this shows that (5.6) holds and completes the proof of the lemma. $\square$

In the next lemma, $q \geq 2$ is the value for which Assumption 2 is satisfied.

LEMMA 5.3. *Suppose b and σ are uniformly bounded. Then there exists a finite constant $C < \infty$ such that for each $\varepsilon > 0$,*

$$\begin{aligned} &\lim_{\triangle(\pi^n) \to 0} \mathbb{P}\left(\sum_{t_i \in \pi^n} \sum_{k=0}^\infty |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1}) > \frac{\delta}{2}\right) \\ &\qquad \leq \begin{cases} \dfrac{C}{\delta} \mathbb{E}\left[\displaystyle\int_0^T \sum_{k=0}^\infty \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon]}(t)\, dt\right], & \textit{if } q = 2, \\ 0, & \textit{if } q > 2. \end{cases} \end{aligned} \tag{5.9}$$

PROOF. Fix $\varepsilon > 0$. Then by Markov's inequality [whose application is justified by (5.12) when $q > 2$, and by (5.13) when $q = 2$] and the monotone convergence theorem

$$\begin{aligned} &\mathbb{P}\left(\sum_{t_i \in \pi^n} \sum_{k=0}^\infty |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1}) > \frac{\delta}{2}\right) \\ &\qquad \leq \frac{2}{\delta} \sum_{t_i \in \pi^n} \sum_{k=0}^\infty \mathbb{E}[|Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1})]. \end{aligned} \tag{5.10}$$

Recall that $a = \sigma^T \sigma$, and let $\bar{C} > 1$ be an upper bound on $|b|$, $|\sigma|$ and $|a|$. By Assumption 2, the definition (2.4) of X and the elementary inequality

$|x+y|^q \le 2^q(|x|^q+|y|^q)$, there exists $K_T<\infty$ such that for each $t_i \in \pi^n$,

$$\begin{aligned}
&\mathbb{E}[|Y(t_i)-Y(t_{i-1})|^p|\mathcal{F}_{t_{i-1}}] \\
&\quad \le K_T \mathbb{E}\Big[\sup_{u\in[t_{i-1},t_i]} |X(u)-X(t_{i-1})|^q|\mathcal{F}_{t_{i-1}}\Big] \\
&\quad \le 2^q K_T \mathbb{E}\Bigg[\sup_{u\in[t_{i-1},t_i]} \left|\int_{t_{i-1}}^u b(Z(v))\,dv\right|^q \\
&\qquad\qquad + \sup_{u\in[t_{i-1},t_i]} \left|\int_{t_{i-1}}^u \sigma(Z(v))\,dB_v\right|^q \Bigg|\mathcal{F}_{t_{i-1}}\Bigg] \\
&\quad \le 2^q K_T \mathbb{E}\left[\bar{C}^q (t_i-t_{i-1})^q + \left(\frac{q}{q-1}\right)^q \left|\int_{t_{i-1}}^{t_i} \sigma(Z(v))\,dB_v\right|^q \Bigg|\mathcal{F}_{t_{i-1}}\right] \\
&\quad \le 2^q K_T \bar{C}^q (t_i-t_{i-1})^q \\
&\qquad + 2^q K_T \left(\frac{q}{q-1}\right)^q \tilde{K} \mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} |a(Z(v))|\,dv\right)^{q/2} \Bigg|\mathcal{F}_{t_{i-1}}\right],
\end{aligned}$$

where the third inequality holds due to the uniform bound on $b(\cdot)$, the Markov property of Z and Doob's maximal martingale inequality, while the fourth inequality follows, with $\tilde{K}<\infty$ a universal constant, by an application of the martingale moment inequality, which is justified since the uniform boundedness on a ensures that the stochastic integral is a martingale.

Define $\tilde{C} \doteq 2^q K_T[\bar{C}^q \vee (q^q \bar{C}^{q/2}\tilde{K}/(q-1)^q)]$. Using the bound on a, the last inequality shows that for each $t_i \in \pi^n$,

$$\mathbb{E}[|Y(t_i)-Y(t_{i-1})|^p|\mathcal{F}_{t_{i-1}}] \le \tilde{C}[(t_i-t_{i-1})^q + (t_i-t_{i-1})^{q/2}]. \tag{5.11}$$

We now consider two cases. If $q>2$, it follows from (5.11) that, for all sufficiently large n such that $\Delta(\pi^n)<1$,

$$\mathbb{E}[|Y(t_i)-Y(t_{i-1})|^p|\mathcal{F}_{t_{i-1}}] \le 2\tilde{C}\Delta(\pi^n)^{q/2-1}(t_i-t_{i-1}).$$

Multiplying both sides of this inequality by $\mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon)}(t_{i-1})$, which is $\mathcal{F}_{t_{i-1}}$-measurable since τ_k^ε and ξ_k^ε are stopping times, then taking expectations and subsequently summing over $k=0,1,\ldots,$ and $t_i\in\pi^n$, it follows that

$$\sum_{t_i\in\pi^n}\sum_{k=0}^\infty \mathbb{E}[|Y(t_i)-Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon)}(t_{i-1})] \le 2\tilde{C}\Delta(\pi^n)^{q/2-1}T. \tag{5.12}$$

Since $\Delta(\pi^n)^{q/2-1}\to 0$ as $n\to\infty$, combining this with (5.10), we then obtain

$$\lim_{\triangle(\pi^n)\to 0} \mathbb{P}\left(\sum_{t_i\in\pi^n}\sum_{k=0}^\infty |Y(t_i)-Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon)}(t_{i-1}) > \frac{\delta}{2}\right) = 0.$$

On the other hand, if $q=2$, again multiplying both sides of (5.11) by $\mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon)}(t_{i-1})$, then taking expectations, subsequently summing over $k=0,1,\dots$, and $t_i\in\pi^n$, and then using the monotone convergence theorem to interchange expectation and summation, we obtain

$$\sum_{t_i\in\pi^n}\sum_{k=0}^{\infty}\mathbb{E}[|Y(t_i)-Y(t_{i-1})|^p\mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon)}(t_{i-1})]$$
$$\le\tilde{C}\left(\Delta(\pi^n)^q+\mathbb{E}\left[\sum_{t_i\in\pi^n}(t_i-t_{i-1})\sum_{k=0}^{\infty}\mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon)}(t_{i-1})\right]\right)<\infty.$$

Sending $\Delta(\pi^n)\to 0$ on both sides of this inequality and invoking the bounded convergence theorem, the right-continuity of $\mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon)}(\cdot)$ and the definition of the Riemann integral, we obtain

$$\lim_{\Delta(\pi^n)\to 0}\sum_{t_i\in\pi^n}\sum_{k=0}^{\infty}\mathbb{E}[|Y(t_i)-Y(t_{i-1})|^p\mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon)}(t_{i-1})]$$
$$\le\tilde{C}\mathbb{E}\left[\int_0^T\sum_{k=0}^{\infty}\mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon)}(t)\,dt\right].$$

Together with (5.10), this shows that (5.9) holds with $C=2\tilde{C}$. $\square$

PROOF OF THEOREM 3.5. Due to Lemma 5.1, using a localization argument and the local boundedness of b and σ stated in Assumption 3, we can assume without loss of generality that a,b and σ are bounded. Then, combining (5.5) with Lemmas 5.2 and 5.3, we have

$$\lim_{\Delta(\pi^n)\to 0}\mathbb{P}\left(\sum_{t_i\in\pi^n}|Y(t_i)-Y(t_{i-1})|^p>\delta\right)$$
$$\le\begin{cases}\dfrac{C}{\delta}\mathbb{E}\left[\displaystyle\int_0^T\sum_{k=0}^{\infty}\mathbb{I}_{[\xi_k^\varepsilon,\tau_{k+1}^\varepsilon]}(t)\,dt\right], & \text{if } q=2,\\ 0, & \text{if } q>2,\end{cases}$$

for every $\varepsilon>0$, and so (5.1) holds for the case $q>2$. If $q=2$, sending $\varepsilon\downarrow 0$ and using the bounded convergence theorem and the definition of the stopping times ξ_k^ε and τ_k^ε, we see that the term on the right-hand side converges to

$$\frac{C}{\delta}\mathbb{E}\left[\int_0^T\mathbb{I}_{\mathcal{V}}(Z(t))\,dt\right]=0,$$

where the last equality follows from (2.5) and the fact that $\mathcal{V}\subset\partial G$. This proves (5.1), and Theorem 3.5 then follows from the discussion at the beginning of Section 5. $\square$

5.3. *Proof of Corollary 3.7.* Suppose that, as in (3.3), the functions L and R on $[0,\infty)$ are given by $L(y) = -c_L y^{\alpha_L}$ and $R(y) = c_R y^{\alpha_R}$ for some $\alpha_L, \alpha_R, c_L, c_R \in (0,\infty)$. As defined in Section 2.4, let $(G, d(\cdot))$ and $\overline{\Gamma}$ be the associated ESP and ESM, and let $Z = (Z_1, Z_2)$ be the associated two-dimensional RBM: $Z = \overline{\Gamma}(B)$, where $B = (B_1, B_2)$ is a standard two-dimensional Brownian motion. Then Assumptions 1 and 3 are automatically satisfied for this family of reflected diffusions. In order to prove the corollary, it suffices to show that Assumption 2 holds. Indeed, then all the assumptions of Theorem 3.5 are satisfied, and Corollary 3.7 follows as a consequence.

We now recall the representation for $Y \doteq Z - B$ that was obtained in Section 4.3 of [2]. First, note that Z_2 is a one-dimensional RBM on $[0,\infty)$ with the pathwise representation $Z_2 = \Gamma_1(B_2)$, where Γ_1 is the one-dimensional reflection map on $[0,\infty)$. Thus, $Y_2 = \Lambda_2(B_2)$, where $\Lambda_2(\psi) \doteq \Gamma_1(\psi) - \psi$ is given explicitly by

$$\Lambda_2(\psi)(t) \doteq \sup_{0 \le s \le t} [-\psi(s)]^+, \qquad \psi \in \mathcal{C}[0,\infty), t \in [0,\infty). \tag{5.13}$$

(Recall that $\mathcal{C}[0,\infty)$ is the space of continuous functions on $[0,\infty)$, equipped with the topology of uniform convergence on compact sets.) Since Y_2 is a nondecreasing process, it is clearly of finite variation. Therefore, to establish Assumption 2, it suffices to show that the inequality (2.7) holds with Y replaced by Y_1. From Section 4.3 of [2], it follows that pathwise $Z_1 = \bar{\Gamma}_{\ell,r}(B_1)$, where $\bar{\Gamma}_{\ell,r}$ is the ESM whose domain is the time-dependent interval $[l(\cdot), r(\cdot)]$, with $l(t) \doteq L(Z_2(t))$ and $r(t) \doteq R(Z_2(t))$, for $t \in [0,\infty)$. A precise definition of $\bar{\Gamma}_{\ell,r}$ is stated as Definition 2.2 of [2], but for the present purpose it suffices to note that Theorem 2.6 of [2] establishes the explicit representation $\bar{\Gamma}_{\ell,r}(\psi) = \psi - \Xi_{\ell,r}(\psi)$, where for $\psi \in \mathcal{C}[0,\infty)$ such that $\psi(0) \in [\ell(0), r(0)]$, and $t \in [0,\infty)$,

$$\begin{aligned} \Xi_{\ell,r}(\psi)(t) \doteq \max\Big(&\Big[0 \wedge \inf_{u \in [0,t]} (\psi(u) - \ell(u))\Big], \\ &\sup_{s \in [0,t]} \Big[(\psi(s) - r(s)) \wedge \inf_{u \in [s,t]} (\psi(u) - \ell(u))\Big]\Big). \end{aligned} \tag{5.14}$$

Thus, we see that $Y_1 = \Lambda_1(B_1, B_2)$, where Λ_1 is the map from $\mathcal{C}[0,\infty)^2$ to $\mathcal{C}[0,\infty)$ given by

$$\Lambda_1 : (\psi_1, \psi_2) \mapsto -\Xi_{L \circ \Gamma_1(\psi_2), R \circ \Gamma_1(\psi_2)}(\psi_1).$$

From the explicit expression for $\Xi_{\ell,r}$ given in (5.14), it can be easily verified that the map $(\ell, r, \psi) \mapsto -\Xi_{\ell,r}(\psi)$ from $\mathcal{C}[0,\infty)^3$ to $\mathcal{C}[0,\infty)$ is Lipschitz continuous. In addition, it follows from (5.13) that the map $\psi \mapsto \Gamma_1(\psi)$ from $\mathcal{C}[0,\infty)$ to itself is also Lipschitz continuous. If L and R are Hölder continuous with exponent $\alpha = \alpha_L \wedge \alpha_R \in (0,1)$, it follows that the composition

maps $\ell = L \circ \Gamma_1$ and $r = R \circ \Gamma_1$ are also Hölder continuous with exponent α. When combined, the above statements then imply that the map Λ_1 is locally Hölder continuous on $\mathcal{C}[0,\infty)^2$ with exponent α, and so (2.7) holds for any $p \geq 2/\alpha$ with, correspondingly, $q = \alpha p$. On the other hand, if L and R are locally Lipschitz continuous [i.e., if (3.3) is satisfied with $\alpha \geq 1$], then clearly $\bar{\Gamma}$ is also locally Lipschitz continuous, and so (2.7) holds with $p = q = 2$. Thus, the result follows in this case as well (note that, due to the localization result of Section 5.1, it suffices for the ESM to be locally Lipschitz or locally Hölder, that is, Lipschitz continuous or Hölder continuous on paths that lie in a compact set on any finite time interval).

REMARK 5.4. The proof above also shows that when $\alpha < 1$, $Z = B + A$, where A is a process of zero p-variation for every $p > 2/\alpha$. However, this is likely to be a sub-optimal result, since given that Z is a Dirichlet process even when the domain is cusp-like, one would expect that Z would also be a Dirichlet process when the domain is flatter (corresponding to $\alpha > 1$). Indeed, more generally, it would be of interest to determine the lowest p-variation that vanishes for a given α, to better understand the relationship between the "roughness" of the paths of Z and the curvature of the boundary of the domain. Such questions motivate a "rough paths" analysis (see, e.g., [21] and [16]) of reflected stochastic processes.

REMARK 5.5. The above class of reflected diffusions provides one example of a situation where the ESM is locally Hölder continuous, but the (generalized) completely-$\mathcal{S}$ condition does not hold. However, we believe that it should be possible to combine a localization argument of the kind used in [6] and the sufficient condition for Lipschitz continuity of the ESM obtained in Theorem 3.3 of [22] to identify a broad class of piecewise smooth domains and directions of reflection where the generalized completely-$\mathcal{S}$ condition fails to hold, but for which the associated ESM is locally Hölder continuous.

APPENDIX A: ELEMENTARY PROPERTIES OF THE ESP

LEMMA A.1. *If (ϕ, η) is a solution to the ESP $(G, d(\cdot))$ for $\psi \in \mathcal{C}_G[0,\infty)$, then for each $0 \leq s < \infty$, (ϕ^s, η^s) is a solution to the ESP for $\phi(s) + \psi^s$, where $\phi^s(\cdot) \doteq \phi(s + \cdot)$,*

$$\psi^s(\cdot) \doteq \psi(s + \cdot) - \psi(s) \quad \textit{and} \quad \eta^s(\cdot) \doteq \eta(s + \cdot) - \eta(s).$$

Moreover, if the ESM is well-defined and Lipschitz continuous on $\mathcal{C}_G[0,\infty)$ then for every $T < \infty$, there exists $\tilde{K}_T < \infty$ such that for every $0 \leq s < t \leq T + s$,

$$|\eta(t) - \eta(s)| \leq \tilde{K}_T \sup_{u \in [0, t-s]} |\psi(s+u) - \psi(s)|.$$

PROOF. Fix $s \in [0,\infty)$ and a path $\psi \in \mathcal{D}_G[0,\infty)$. The first statement follows from Lemma 2.3 of [22]. It implies that $\eta^s = \bar{\Gamma}(\psi^1) - \psi^1$, where $\psi^1 \doteq \phi(s) + \psi^s$. On the other hand, consider the path ψ^2 which is equal to the constant $\phi(s)$ on $[0,\infty)$, that is, $\psi^2(u) \doteq \phi(s)$ for all $u \in [0,\infty)$. Then clearly $(\phi(s), 0)$ is the unique solution to the ESP for ψ^2, that is, $0 = \bar{\Gamma}(\psi^2)(u) - \psi^2(u)$ for all $u \in [0,\infty)$. Using the Lipschitz continuity of the ESM, for $\delta \in [0, T-s]$ we obtain

$$\begin{aligned}
|\eta^s(\delta) - 0| &\leq \sup_{u\in[0,\delta]} |\bar{\Gamma}(\psi^1)(u) - \psi^1(u) - \bar{\Gamma}(\psi^2)(u) + \psi^2(u)| \\
&\leq \sup_{u\in[0,\delta]} |\bar{\Gamma}(\psi^1)(u) - \bar{\Gamma}(\psi^2)(u)| + \sup_{u\in[0,\delta]} |\psi^1(u) - \psi^2(u)| \\
&\leq K_T \sup_{u\in[0,\delta]} |\psi^s(u)| + \sup_{u\in[0,\delta]} |\psi^s(u)|,
\end{aligned}$$

where $K_T < \infty$ is the Lipschitz constant of $\bar{\Gamma}$ on $[0,T]$. The lemma follows by letting $\tilde{K}_T \doteq K_T + 1$ and $\delta = t - s$. $\square$

APPENDIX B: AUXILIARY RESULTS

For completeness, we provide the proof of the fact that the sequences of times defined in Section 4.1.3 are stopping times.

LEMMA B.1. *$\{\beta_n^\varepsilon\}_{n\in\mathbb{N}}$, $\{\beta_{(k),n}^\varepsilon\}_{n\in\mathbb{N}}$, $k \in \mathbb{N}$, are sequences of $\{\mathcal{F}_t\}$-stopping times. Also, $\{\beta_n^{k,\varepsilon}\}_{n\in\mathbb{N}}$, $k \in \mathbb{N}$, are sequences of $\{\mathcal{F}_t^k\}$-stopping times.*

PROOF. Clearly, $\beta_0^\varepsilon \doteq 0$ is an $\{\mathcal{F}_t\}$-stopping time. Now, suppose β_{n-1}^ε is an $\{\mathcal{F}_t\}$-stopping time and note that for each $\varepsilon > 0$, $n \in \mathbb{N}$ and $t \in [0,\infty)$,

$$\{\beta_n^\varepsilon \leq t\} = \bigcup_{k\in\mathbb{Z}} [\{\beta_{n-1}^\varepsilon \leq t\} \cap \{Z(\beta_{n-1}^\varepsilon) \in H_{2^k\varepsilon}\} \cap A_{k,n}^\varepsilon(t)],$$

where

$$A_{k,n}^\varepsilon(t) \doteq \Big\{ \sup_{s\in[\beta_{n-1}^\varepsilon, t]} \langle Z(s), \vec{\mathbf{v}}\rangle \geq 2^{k+1}\varepsilon \Big\} \cup \Big\{ \inf_{s\in[\beta_{n-1}^\varepsilon, t]} \langle Z(s), \vec{\mathbf{v}}\rangle \leq 2^{k-1}\varepsilon \Big\}.$$

Then $\{\beta_{n-1}^\varepsilon \leq t\} \in \mathcal{F}_t$ because β_{n-1}^ε is an $\{\mathcal{F}_t\}$-stopping time. Since Z is continuous we also know that $\{\beta_{n-1}^\varepsilon \leq t\} \cap \{Z(\beta_{n-1}^\varepsilon) \in H_{2^k\varepsilon}\}$ lies in $\mathcal{F}_t$. In addition, the continuity of $\langle Z, \vec{\mathbf{v}}\rangle$ and the fact that $[2^{k+1}\varepsilon, \infty)$ and $(-\infty, 2^{k-1}\varepsilon]$ are closed show that $\{\beta_{n-1}^\varepsilon \leq t\} \cap A_{n,k}^\varepsilon(t) \in \mathcal{F}_t$. When combined, this implies that $\{\beta_n^\varepsilon \leq t\} \in \mathcal{F}_t$ or, equivalently, that β_n^ε is an $\{\mathcal{F}_t\}$-stopping time, and the first assertion follows by induction. The proof for the other sequences is exactly analogous. $\square$

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